Index theory of uniform pseudodifferential operators

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Abstract

We generalize Roe's index theorem for graded generalized Dirac operators on amenable manifolds to multigraded elliptic uniform pseudodifferential operators.

This generalization will follow as a corollary from a local index theorem that is valid on any manifold of bounded geometry. This local formula incorporates the uniform estimates that are present in the definition of our class of pseudodifferential operators which is more general than similar classes defined by other authors.

We will revisit Špakula's uniform K-homology and show that multigraded elliptic uniform pseudodifferential operators naturally define classes in it. For this we will investigate uniform K-homology more closely, e.g., construct the external product and show invariance under weak homotopies. The latter will be used to refine and extend Špakula's results about the rough Baum–Connes assembly map.

We will identify the dual theory of uniform K-homology. We will give a simple definition of uniform K-theory for all metric spaces and in the case of manifolds of bounded geometry we will give an interpretation of it via vector bundles of bounded geometry. Using a version of Mayer–Vietoris induction that is adapted to our needs, we will prove Poincaré duality between uniform K-theory and uniform K-homology for spin^c manifolds of bounded geometry.

We will construct Chern characters from uniform K-theory to bounded de Rham cohomology and from uniform K-homology to uniform de Rham homology. Using the adapted Mayer–Vietoris induction we will also show that these Chern characters induce isomorphisms modulo torsion.

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Contents

1 Introduction

Recall the following index theorem of Roe for amenable manifolds (with notation adapted to the one used in this article).

Theorem ([Roe88a, Theorem 8.2])**.** *Let M be a Riemannian manifold of bounded geometry and D a generalized Dirac operator associated to a graded Dirac bundle S of bounded geometry over M.*

Let $(M_i)_i$ be a Følner sequence[1] *for M, $\tau \in (\ell^\infty)^*$ a linear functional associated to a free ultrafilter on $\mathbb{N}$, and θ the corresponding trace on the uniform Roe algebra of M.*

Then we have

$$\theta(\mu_u(D)) = \tau\Big(\frac{1}{\operatorname{vol} M_i} \int_{M_i} \operatorname{ind}(D)\Big).$$

Here $\operatorname{ind}(D)$ is the usual integrand for the topological index of D in the Atiyah–Singer index formula, so the right hand side is topological in nature. On the left hand side of the formula we have the coarse index class $\mu_u(D) \in K_0(C_u^*(M))$ of D in the K-theory of the uniform Roe algebra of M evaluated under the trace θ. This is an analytic expression and may be computed as $\theta(\mu_u(D)) = \tau\Big(\frac{1}{\operatorname{vol} M_i} \int_{M_i} \operatorname{tr}_s k_{f(D)}(x,x)\, dx\Big)$, where $k_{f(D)}(x,y)$ is the integral kernel of the smoothing operator $f(D)$, where f is an even Schwartz function with $f(0) = 1$.

In this article we will generalize this theorem to multigraded, elliptic, symmetric uniform pseudodifferential operators. So especially we also encompass Toeplitz operators since they are included in the ungraded case. This generalization will follow from a local index theorem that will hold on any manifold of bounded geometry, i.e., without an amenability assumption on M.

Let us state our local index theorem in the formulation using twisted Dirac operators associated to spin^c structures.

Theorem. *Let M be an m-dimensional spin^c manifold of bounded geometry and without boundary. Denote the associated Dirac operator by D.*

Then we have the following commutative diagram:

$$\begin{tikzcd} K_u^*(M) \arrow[r, "\cdot\cap[D]", "\cong"'] \arrow[d, "\operatorname{ch}(\cdot)\wedge\operatorname{ind}(D)"'] & K^u_{m-*}(M) \arrow[d, "\alpha_*\circ\operatorname{ch}^*"] \\ H^*_{b,\mathrm{dR}}(M) \arrow[r, "\cong"'] & H^{u,\mathrm{dR}}_{m-*}(M) \end{tikzcd}$$

Here $K^u_{m-*}(M)$ is the uniform K-homology of M invented by Špakula in [Špa09] and $K_u^*(M)$ the corresponding uniform K-theory which we will define in Section 4. The map $\cdot \cap [D]$ is the cap product and that it is an isomorphism will be shown in Section 4.4. Moreover, $H^*_{b,\mathrm{dR}}(M)$ denotes the bounded de Rham cohomology of M and $\operatorname{ind}(D)$ the

[1]That is to say, for every $r > 0$ we have $\frac{\operatorname{vol} B_r(\partial M_i)}{\operatorname{vol} M_i} \xrightarrow{i\to\infty} 0$. Manifolds admitting such a sequence are called *amenable*.

topological index class of D in there. Furthermore, $H^{u,\mathrm{dR}}_{m-*}(M)$ is the uniform de Rham homology of M to be defined in Section 5.2 via Connes' cyclic cohomology, and that it is Poincaré dual to bounded de Rham cohomology is stated in Theorem 5.9. Finally, let us note that we will also prove in Section 5.3 that the Chern characters induce isomorphisms modulo torsion, similar to the case of compact manifolds.

Using a series of steps as in the proof of [CM90, Theorem 3.9] we will be able to generalize the above computation of the Poincaré dual of $(\alpha_* \circ \mathrm{ch}^*)([D]) \in H^{u,\mathrm{dR}}_{m-*}(M)$ to symmetric and elliptic uniform pseudodifferential operators.

Theorem. *Let M be a Riemannian manifold of bounded geometry and without boundary and let P be a symmetric and elliptic uniform pseudodifferential operator.*

Then $\mathrm{ind}(P) \in H^*_{b,\mathrm{dR}}(M)$ is mapped by the duality map $H^*_{b,\mathrm{dR}}(M) \to H^{u,\mathrm{dR}}_*(M)$ to the class $(\alpha_* \circ \mathrm{ch}^*)([P]) \in H^{u,\mathrm{dR}}_*(M)$.

Using the above local index theorem we will derive as a corollary the following local index formula:

Corollary. *Let $[\varphi] \in H^k_{c,\mathrm{dR}}(M)$ be a compactly supported cohomology class and define the analytic index $\mathrm{ind}_{[\varphi]}(P)$ as in [CM90] for P a multigraded, elliptic, symmetric uniform pseudodifferential operator of positive order. Then we have*

$$\mathrm{ind}_{[\varphi]}(P) = \int_M \mathrm{ind}(P) \wedge [\varphi]$$

and this pairing is continuous, i.e., $\int_M \mathrm{ind}(P) \wedge [\varphi] \le \|\mathrm{ind}(P)\|_\infty \cdot \|[\varphi]\|_1$, where $\|\cdot\|_\infty$ denotes the sup-seminorm on $H^{m-k}_{b,\mathrm{dR}}(M)$ and $\|\cdot\|_1$ the L^1-seminorm on $H^k_{c,\mathrm{dR}}(M)$.

Note that the corollary reads basically the same as the local index formula of Connes and Moscovici from [CM90]. The fundamentally new thing in it is the continuity statement for which we need the uniformity assumption for P.

As a second corollary to the above local index theorem we will, as already written, derive the generalization of Roe's index theorem for amenable manifolds.

Corollary. *Let M be a manifold of bounded geometry and without boundary, let $(M_i)_i$ be a Følner sequence for M and let $\tau \in (\ell^\infty)^*$ be a linear functional associated to a free ultrafilter on $\mathbb{N}$. Denote the from the choice of Følner sequence and functional τ resulting functional on $K_0(C^*_u(M))$ by θ.*

Then for both $p \in \{0,1\}$, every $[P] \in K^u_p(M)$ for P a p-graded, elliptic, symmetric uniform pseudodifferential operator over M, and every $u \in K^p_u(M)$ we have

$$\langle u, [P]\rangle_\theta = \langle \mathrm{ch}(u) \wedge \mathrm{ind}(P), [M]\rangle_{(M_i)_i,\tau}.$$

Roe's theorem from [Roe88a] is the special case where $P = D$ is a graded (i.e., $p = 0$) Dirac operator and $u = [\mathbb{C}]$ is the class in $K^0_u(M)$ of the trivial, 1-dimensional vector bundle over M.

Note that this global index theorem arising from a Følner sequence is just a special case of a certain rough index theory, where one pairs classes from the so-called rough

cohomology with classes in the K-theory of the uniform Roe algebra, and Følner sequences give naturally classes in this rough cohomology. For details see the thesis [Mav95] of Mavra. It seems that it must be possible to combine the above local index theorem with this rough index theory, since it is possible in the special case of Følner sequences. The author will investigate this in a forthcoming article.

Let us say a few words about the proof of the above index theorem for uniform pseudodifferential operators. Roe used in [Roe88a] the heat kernel method to prove his index theorem for amenable manifolds and therefore, since the heat kernel method does only work for Dirac operators, it can not encompass uniform pseudodifferential operators. So what we will basically do in this paper is to set up all the necessary theory in order to be able to reduce the index problem from pseudodifferential operators to Dirac operators.

As it turns out, the only useful definition of pseudodifferential operators on non-compact manifolds is the uniform one since otherwise we can not guarantee that the operators do have continuous extensions to Sobolev spaces (we will elaborate more on this at the beginning of the next Section 2). Now in the reduction of the index problem from pseudodifferential operators to Dirac operators one can use, e.g., the fact that for spin^c manifolds we have K-Poincaré duality between K-theory and K-homology (i.e., every class of an abstractly elliptic operator may be represented by a difference of twisted Dirac operators). In order to do the same here in our uniform case we therefore first need uniform K-theory and K-homology theories (since usual K-homology does not consider at all any uniformity that we might have for our operators, and since we are forced to work with uniform pseudodifferential operators, it is quite clear that we also need a new K-homology theory that incorporates uniformity). To our luck, Špakula already did this for us, i.e., he already defined in [Špa09] a uniform K-homology theory and it turns out that this is exactly what we need. Evidence for the latter statement is provided by the fact that Špakula constructed a rough assembly map from uniform K-homology to the K-theory of the uniform Roe algebra, and Roe uses in [Roe88a] the latter groups as receptacles for the analytic index classes for his index theorem.

So from the above it is quite clear what we have to do: After defining and investigating the class of uniform pseudodifferential operators, that we are interested in, we have to look at uniform K-homology more closely (i.e., Špakula did not construct the Kasparov product for it, but we need it crucially to show homotopy invariance of uniform K-homology and therefore we will have to do this construction here). Then we have to identify the corresponding dual theory to uniform K-homology and prove the uniform K-Poincaré duality theorem. With this at our disposal we will then be able to reduce the uniform index problem for uniform pseudodifferential operators to Dirac operators (for this we will also have to prove a uniform version of the Thom isomorphism in order to be able to conclude that symbol classes of uniform pseudodifferential operators may be represented by symbol classes of Dirac operators). So it remains to show the uniform index theorem for Dirac operators. But since up to this point we will already have set up all the needed machinery, its proof will be basically the same as the proof of the local index theorem of Connes and Moscovici in [CM90].

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2 Pseudodifferential operators on open manifolds

Let us explain why on non-compact manifolds we necessarily have to look at uniform pseudodifferential operators. Recall that on $\mathbb{R}^m$ an operator P is called pseudodifferential, if it is given by

$$(Pu)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^m} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)\ d\xi,$$

where $\hat{u}$ denotes the Fourier transform of u and the function $p(x,\xi)$ satisfies for some $k \in \mathbb{Z}$ the estimates $\|D_x^\alpha D_\xi^\beta p(x,\xi)\| \le C^{\alpha\beta}(1+|\xi|)^{k-|\beta|}$ for all multi-indices α and β. On manifolds one calls an operator pseudodifferential if one has the above representation in any local chart. But if the manifold is not compact, we get the problem that this is not sufficient to guarantee that the operator has continuous extensions to Sobolev spaces.[2] For this we additionally have to require that the above bounds $C^{\alpha\beta}$ are uniform across all the local charts. But since this is not well-defined (choosing a different atlas may distort the bounds arbitrarily large across the charts of the atlas), we will have to restrict the charts to exponential charts and additionally we will have to assume that our manifold has bounded geometry (this restrictions become clear when one looks at Lemma 2.3).

[2] We are ignoring in this discussion the fact that on non-compact manifolds we also need a condition on the behaviour of the integral kernel of P at infinity. So assume for the moment that P has finite propagation. Our final definition will require P to be quasilocal at infinity (see Section 2.2).

The local definition of uniform pseudodifferential operators that we investigate was already given by Kordyukov in [Kor91], by Shubin in [Shu92] and by Taylor in [Tay08]. And these are, to the surprise of the author, the only three instances that the author is aware of and where such uniform pseudodifferential operators were investigated. Regarding the control of the integral kernels of these operators at infinity, Kordyukov and Shubin impose that their operators must have finite propagation, i.e., that there is an $R > 0$ such that the integral kernel $k(x, y)$ of the pseudodifferential operator vanishes for all x, y with $d(x, y) > R$ (note that pseudodifferential operators always have an integral kernel that is smooth outside the diagonal). Taylor requires more generally an exponential decay of the integral kernel at infinity, and for some results this decay should be faster than the volume growth of the manifold. We require that our pseudodifferential operators are quasilocal[3]. So at the end the definition of pseudodifferential operators on open manifold that we give is novel and the most general one that the author is aware of.

Let us explain why we want our operators to be quasilocal, i.e., why the restrictions on the integral kernel of the other authors is not good enough for us: Recall that in order to compute Roe's analytic index of an operator D of Dirac type, we have to consider the operator $f(D)$, where f is an even Schwartz function with $f(0) = 1$. Now usually $f(D)$ will not have finite propagation, but it will be a quasilocal operator. This was proven by Roe and we will generalize this crucial fact to pseudodifferential operators. So this means that we stay in our class of operators when computing analytic indices, but this is not true if we would work with the definitions of the other authors. Note that the proof of the fact that $f(P)$ is quasilocal requires substantial analysis and is one of our key technical lemmas.

2.1 Bounded geometry

We will define in this section the notion of bounded geometry for manifolds and for vector bundles and discuss basic facts about uniform C^r-spaces and Sobolev spaces on them.

Definition 2.1. We will say that a Riemannian manifold M has *bounded geometry*, if

- the curvature tensor and all its derivatives are bounded, i.e., $\|\nabla^k \operatorname{Rm}(x)\| < C_k$ for all $x \in M$ and $k \in \mathbb{N}_0$, and
- the injectivity radius is uniformly positive, i.e., $\text{inj-rad}_M(x) > \varepsilon > 0$ for all points $x \in M$ and for a fixed $\varepsilon > 0$.

If $E \to M$ is a vector bundle with a metric and compatible connection, we say that E *has bounded geometry*, if the curvature tensor of E and all its derivatives are bounded.

Examples 2.2. There are plenty of examples of manifolds of bounded geometry. The most important ones are coverings of compact Riemannian manifolds equipped with

[3] An operator $A\colon H^r(E) \to H^s(F)$ is *quasilocal*, if there is some function $\mu\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ with $\mu(R) \to 0$ for $R \to \infty$ and such that for all $L \subset M$ and all $u \in H^r(E)$ with $\operatorname{supp} u \subset L$ we have $\|Au\|_{H^s, M - B_R(L)} \leq \mu(R) \cdot \|u\|_{H^r}$.

the pull-back metric, homogeneous manifolds with an invariant metric, and leafs in a foliation of a compact Riemannian manifold (this is proved by Greene in [Gre78, lemma on page 91 and the paragraph thereafter]).

For vector bundles, the most important examples are of course again pull-back bundles of bundles over compact manifolds equipped with the pull-back metric and connection, and the tangent bundle of a manifold of bounded geometry.

Furthermore, if E and F are two vector bundles of bounded geometry, then the dual bundle E^*, the direct sum $E \oplus F$, the tensor product $E \otimes F$ (and so especially also the homomorphism bundle $\operatorname{Hom}(E, F) = F \otimes E^*$) and all exterior powers $\Lambda^l E$ are also of bounded geometry. If E is defined over M and F over N, then their external tensor product[4] $E \boxtimes F$ over $M \times N$ is also of bounded geometry.

Greene proved in [Gre78, Theorem 2'] that there are no obstructions against admitting a metric of bounded geometry, i.e., every smooth manifold without boundary admits one. On manifolds of bounded geometry there is also no obstruction for a vector bundle to admit a metric and compatible connection of bounded geometry. The proof (i.e., the construction of the metric and the connection) is done in a uniform covering of M by normal coordinate charts and subordinate uniform partition of unity (we will discuss these things in a moment) and we have to use the local characterization of bounded geometry for vector bundles from Lemma 2.5.

We will now state an important characterization in local coordinates of bounded geometry since it will allow us to show that certain local definitions (like the one of pseudodifferential operators) are independent of the chosen normal coordinate charts.

Lemma 2.3 ([Shu92, Appendix A1.1])**.** *Let the injectivity radius of M be positive.*

Then the curvature tensor of M and all its derivatives are bounded if and only if for any $0 < r < \text{inj-rad}_M$ all the transition functions between overlapping normal coordinate charts of radius r are uniformly bounded, as are all their derivatives (i.e., the bounds can be chosen to be the same for all transition functions).

Another fact which we will need about manifolds of bounded geometry is the existence of uniform covers by normal coordinate charts and corresponding partitions of unity. A proof may be found in, e.g., [Shu92, Appendix A1.1] (Shubin addresses the first statement about the existence of such covers actually to the paper [Gro81a] of Gromov).

Lemma 2.4. *Let M be a manifold of bounded geometry.*

For every $0 < \varepsilon < \frac{\text{inj-rad}_M}{3}$ there exists a covering of M by normal coordinate charts of radius ε with the properties that the midpoints of the charts form a uniformly discrete set in M and that the coordinate charts with double radius 2ε form a uniformly locally finite cover of M.

Furthermore, there is a subordinate partition of unity $1 = \sum_i \varphi_i$ with $\operatorname{supp} \varphi_i \subset B_{2\varepsilon}(x_i)$, such that in normal coordinates the functions φ_i and all their derivatives are uniformly bounded (i.e., the bounds do not depend on i).

[4] The fiber of $E \boxtimes F$ over the point $(x, y) \in M \times N$ is given by $E_x \otimes F_y$.

If the manifold M has bounded geometry, we have analogous equivalent local characterizations of bounded geometry for vector bundles as for manifolds. The equivalence of the first two bullet points in the next lemma is stated in, e.g., [Roe88a, Proposition 2.5]. Concerning the third bullet point, the author could not find any citable reference in the literature (though Shubin uses in [Shu92] this as the actual definition).

Lemma 2.5. *Let M be a manifold of bounded geometry and $E \to M$ a vector bundle. Then the following are equivalent:*

- *E has bounded geometry,*
- *the Christoffel symbols $\Gamma^{\beta}_{i\alpha}(y)$ of E with respect to synchronous framings (considered as functions on the domain B of normal coordinates at all points) are bounded, as are all their derivatives, and this bounds are independent of $x \in M$, $y \in \exp_x(B)$ and i, α, β, and*
- *the matrix transition functions between overlapping synchronous framings are uniformly bounded, as are all their derivatives (i.e., the bounds are the same for all transition functions).*

We will now give the definition of uniform C^∞-spaces together with a local characterization on manifolds of bounded geometry. The interested reader is refered to, e.g., the papers [Roe88a, Section 2] or [Shu92, Appendix A1.1] of Roe and Shubin for more information regarding these uniform C^∞-spaces.

Definition 2.6 (C^r-bounded functions)**.** Let $f \in C^\infty(M)$. We will say that f is a *C^r_b-function*, or equivalently that it is *C^r-bounded*, if $\|\nabla^i f\|_\infty < C_i$ for all $0 \le i \le r$.

If M has bounded geometry, being C^r-bounded is equivalent to the statement that in every normal coordinate chart $|\partial^\alpha f(y)| < C_\alpha$ for every multiindex α with $|\alpha| \le r$ (where the constants C_α are independent of the chart).

Of course, the definition of C^r-boundedness and its equivalent characterization in normal coordinate charts for manifolds of bounded geometry make also sense for sections of vector bundles of bounded geometry (and so especially also for vector fields, differential forms and other tensor fields).

Definition 2.7 (Uniform C^∞-spaces)**.** Let E be a vector bundle of bounded geometry over M. We will denote the *uniform C^r-space* of all C^r-bounded sections of E by $C^r_b(E)$.

Furthermore, we define the *uniform C^∞-space $C^\infty_b(E)$* as the Fréchet space

$$C^\infty_b(E) := \bigcap_r C^r_b(E).$$

Now we get to Sobolev spaces on manifolds of bounded geometry. Much of the following material is from [Shu92, Appendix A1.1] and [Roe88a, Section 2], where an interested reader can find more thorough discussions of this matters.

Let $s \in C_c^\infty(E)$ be a compactly supported, smooth section of some vector bundle $E \to M$ with metric and connection ∇. For $k \in \mathbb{N}_0$ and $p \in [1, \infty)$ we define the global $W^{k,p}$-Sobolev norm of s by

$$\|s\|_{W^{k,p}}^p := \sum_{i=0}^{k} \int_M \|\nabla^i s(x)\|^p dx. \tag{2.1}$$

Definition 2.8 (Sobolev spaces $W^{k,p}(E)$)**.** Let E be a vector bundle which is equipped with a metric and a connection. The *$W^{k,p}$-Sobolev space of E* is the completion of $C_c^\infty(E)$ in the norm $\|\cdot\|_{W^{k,p}}$ and will be denoted by $W^{k,p}(E)$.

If E and M^m both have bounded geometry than the Sobolev norm (2.1) is equivalent to the local one given by

$$\|s\|_{W^{k,p}}^p \overset{\text{equiv}}{=} \sum_{i=1}^{\infty} \|\varphi_i s\|_{W^{k,p}(B_{2\varepsilon}(x_i))}^p, \tag{2.2}$$

where the balls $B_{2\varepsilon}(x_i)$ and the subordinate partition of unity φ_i are as in Lemma 2.4, we have chosen synchronous framings and $\|\cdot\|_{W^{k,p}(B_{2\varepsilon}(x_i))}$ denotes the usual Sobolev norm on $B_{2\varepsilon}(x_i) \subset \mathbb{R}^m$. This equivalence enables us to define the Sobolev norms for all $k \in \mathbb{R}$.

Assuming bounded geometry, the usual embedding theorems are true:

Theorem 2.9 ([Aub98, Theorem 2.21])**.** *Let E be a vector bundle of bounded geometry over a manifold M^m of bounded geometry and without boundary.*

Then we have for all values $(k - r)/n > 1/p$ continuous embeddings

$$W^{k,p}(E) \subset C_b^r(E).$$

We define the space

$$W^{\infty,p}(E) := \bigcap_{k \in \mathbb{N}} W^{k,p}(E)$$

and equip it with the obvious Fréchet topology. The Sobolev Embedding Theorem tells us now that we have for all p a continuous embedding

$$W^{\infty,p}(E) \hookrightarrow C_b^\infty(E).$$

For $p = 2$ we will write $H^k(E)$ for $W^{k,2}(E)$. This are Hilbert spaces and for $k < 0$ the space $H^k(E)$ coincides with the dual of $H^{-k}(E)$, regarded as a space of distributional sections of E.

We will now investigate the Sobolev spaces $H^\infty(E)$ and $H^{-\infty}(E)$ of infinite orders. They are crucial since they will allow us to define smoothing operators and hence the important algebra $\mathcal{U}_{-\infty}^*(E)$ in the next section.

Lemma 2.10. *The topological dual of $H^\infty(E)$ is given by*

$$H^{-\infty}(E) := \bigcup_{k \in \mathbb{N}} H^{-k}(E).$$

Let us equip the space $H^{-\infty}(E)$ with the locally convex topology defined as follows: the Fréchet space $H^\infty(E) = \varprojlim H^k(E)$ is the projective limit of the Banach spaces $H^k(E)$, so using dualization we may put on the space $H^{-\infty}(E)$ the inductive limit topology denoted $\iota(H^{-\infty}(E), H^\infty(E))$:

$$H_\iota^{-\infty}(E) := \varinjlim H^{-k}(E).$$

It enjoys the following universal property: a linear map $A\colon H_\iota^{-\infty}(E) \to F$ to a locally convex topological vector space F is continuous if and only if $A|_{H^{-k}(E)}\colon H^{-k}(E) \to F$ is continuous for all $k \in \mathbb{N}$.

Later we will need to know how the bounded[5] subsets of $H_\iota^{-\infty}(E)$ look like, which is the content of the following lemma. In its proof we will also deduce, solely for the enjoyment of the reader, some nice properties of the spaces $H^\infty(E)$ and $H_\iota^{-\infty}(E)$.

Lemma 2.11. *The space $H_\iota^{-\infty}(E) := \varinjlim H^{-k}(E)$ is a* regular inductive limit*, i.e., for every bounded subset $B \subset H_\iota^{-\infty}(E)$ exists some $k \in \mathbb{N}$ such that B is already contained in $H^{-k}(E)$ and bounded there.*[6]

Proof. Since all $H^{-k}(E)$ are Fréchet spaces, we may apply the following corollary of Grothendieck's Factorization Theorem: the inductive limit $H_\iota^{-\infty}(E)$ is regular if and only if it is locally complete (see, e.g., [PCB87, Lemma 7.3.3(i)]). To avoid introducing more burdensome vocabulary, we won't define the notion of local completeness here since we will show something stronger: $H_\iota^{-\infty}(E)$ is actually complete[7].

From [BB03, Sections 3.(a & b)] we conclude the following: since each $H^k(E)$ is a Hilbert space, the Fréchet space $H^\infty(E)$ is the projective limit of reflexive Banach spaces and therefore totally reflexive[8]. It follows that $H^\infty(E)$ is distinguished, which can be characterized by $H_\beta^{-\infty}(E) = H_\iota^{-\infty}(E)$, where $\beta(H^{-\infty}(E), H^\infty(E))$ is the strong topology on $H^{-\infty}(E)$. Now without defining the strong topology we just note that strong dual spaces of Fréchet space are always complete. ☐

2.2 Quasilocal smoothing operators

We will discuss in this section the definition and basic properties of smoothing operators on manifolds of bounded geometry and we will introduce the notion of quasilocal operators. The quasilocal smoothing operators will be the $(-\infty)$-part of our uniform pseudodifferential operators that we are going to define in the next section.

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[5]A subset $B \subset H_\iota^{-\infty}(E)$ is *bounded* if and only if for all open neighbourhoods $U \subset H_\iota^{-\infty}(E)$ of 0 there exists $\lambda > 0$ with $B \subset \lambda U$.

[6]Note that the converse does always hold for inductive limits, i.e., if $B \subset H^{-k}(E)$ is bounded, then it is also bounded in $H_\iota^{-\infty}(E)$.

[7]That is to say, every Cauchy net converges. In locally convex spaces, being Cauchy and to converge is meant with respect to each of the seminorms simultaneously.

[8]That is to say, every quotient of it is reflexive, i.e., the canonical embeddings of the quotients into their strong biduals are isomorphisms of topological vector spaces.

Definition 2.12 (Smoothing operators)**.** Let M be a manifold of bounded geometry and E and F two vector bundles of bounded geometry over M. We will call a continuous linear operator $A\colon H^{-\infty}_\iota(E) \to H^\infty(F)$ a *smoothing operator*.

Lemma 2.13. *A linear operator $A\colon H^{-\infty}_\iota(E) \to H^\infty(F)$ is continuous if and only if it is bounded as an operator $H^{-k}(E) \to H^l(F)$ for all $k, l \in \mathbb{N}$.*

Let us denote by $\mathfrak{B}(H^{-\infty}_\iota(E), H^\infty(E))$ the algebra of all smoothing operators from E to itself. Due to the above lemma we may equip it with the countable family of norms $(\|\cdot\|_{-k,l})_{k,l\in\mathbb{N}}$ so that it becomes a Fréchet space[9].

Now let us get to the main property of smoothing operators that we will need, namely that they can be represented as integral operators with a uniformly bounded smooth kernel. Let $A\colon H^{-\infty}_\iota(E) \to H^\infty(F)$ be given. Then we get by the Sobolev Embedding Theorem 2.9 a continuous operator $A\colon H^{-\infty}_\iota(E) \to C^\infty_b(F)$ and so may conclude by the Schwartz Kernel Theorem for regularizing operators[10] that A has a smooth integral kernel $k_A \in C^\infty(F \boxtimes E^*)$, which is uniformly bounded as are all its derivatives, because of the bounded geometry of M and the vector bundles E and F, i.e., $k_A \in C^\infty_b(F \boxtimes E^*)$.

From the proof of the Schwartz Kernel Theorem for regularizing operators we also see that the assignment of the kernel to the operator is continuous against the Fréchet topology on $\mathfrak{B}(H^{-\infty}_\iota(E), H^\infty(F))$. Furthermore, due to Lemma 2.11 this topology coincides with the topology of bounded convergence[11] on $\mathfrak{B}(H^{-\infty}_\iota(E), H^\infty(F))$. Note that we need this equality of topologies to cite [Roe88a, Proposition 2.9] for the next proposition, i.e., so that our wording of it coincides with the wording in the cited proposition.

Proposition 2.14 ([Roe88a, Proposition 2.9])**.** *Let $A\colon H^{-\infty}_\iota(E) \to H^\infty(F)$ be a smoothing operator. Then A can be written as an integral operator with kernel $k_A \in C^\infty_b(F \boxtimes E^*)$. Furthermore, the map*

$$\mathfrak{B}(H^{-\infty}_\iota(E), H^\infty(F)) \to C^\infty_b(F \boxtimes E^*)$$

associating a smoothing operator its kernel is continuous.

Let $L \subset M$ be any subset. We will denote by $\|\cdot\|_{H^r,L}$ the seminorm on the Sobolev space $H^r(E)$ given by

$$\|u\|_{H^r,L} := \inf\{\|u'\|_{H^r} \mid u' \in H^r(E), u' = u \text{ on a neighbourhood of } L\}.$$

[9]That is to say, a topological vector space whose topology is Hausdorff and induced by a countable family of seminorms such that it is complete with respect to this family of seminorms.

[10]Note that the usual wording of the Schwartz Kernel Theorem for regularizing operators requires the domain $H^{-\infty}(E)$ to be equipped with the weak-* topology $\sigma(H^{-\infty}(E), H^\infty(F))$ and A to be continuous against it. But one actually only needs the domain to be equipped with the inductive limit topology. To see this, one can look at the proof of the Schwartz Kernel Theorem for regularizing kernels as in, e.g., [Gan10, Theorem 3.18].

[11]A basis of neighbourhoods of zero for the topology of bounded convergence is given by the subsets $M(B, U) \subset \mathfrak{B}(H^{-\infty}_\iota(E), H^\infty(F))$ of all operators T with $T(B) \subset U$, where B ranges over all bounded subsets of $H^{-\infty}_\iota(E)$ and U over a basis of neighbourhoods of zero in $H^\infty(F)$.

Definition 2.15 (Quasilocal operators, [Roe88a, Section 5])**.** We will call a continuous operator $A\colon H^r(E) \to H^s(F)$ *quasilocal*, if there is a function $\mu\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ with $\mu(R) \to 0$ for $R \to \infty$ and such that for all $L \subset M$ and all $u \in H^r(E)$ with $\operatorname{supp} u \subset L$ we have

$$\|Au\|_{H^s, M - B_R(L)} \leq \mu(R) \cdot \|u\|_{H^r}.$$

Such a function μ will be called a *dominating function for A*.

We will say that an operator $A\colon C_c^\infty(E) \to C^\infty(F)$ is a *quasilocal operator of order k*[12] for some $k \in \mathbb{Z}$, if A has a continuous extension to a quasilocal operator $H^s(E) \to H^{s-k}(F)$ for all $s \in \mathbb{Z}$.

A smoothing operator $A\colon H_\iota^{-\infty}(E) \to H^\infty(F)$ will be called *quasilocal*, if A is quasilocal as an operator $H^{-k}(E) \to H^l(F)$ for all $k, l \in \mathbb{N}$ (from which it follows that A is also quasilocal for all $k, l \in \mathbb{Z}$).

If we regard a smoothing operator A as an operator $L^2(E) \to L^2(F)$, we get a uniquely defined adjoint $A^*\colon L^2(F) \to L^2(E)$. Its integral kernel will be given by

$$k_{A^*}(x,y) := k_A(y,x)^* \in C_b^\infty(E \boxtimes F^*),$$

where $k_A(y,x)^* \in F_y^* \otimes E_x$ is the dual element of $k_A(y,x) \in F_y \otimes E_x^*$.

Definition 2.16 (cf. [Roe88a, Definition 5.3])**.** We will denote the set of all quasilocal smoothing operators $A\colon H_\iota^{-\infty}(E) \to H^\infty(F)$ with the property that their adjoint operator A^* is also a quasilocal smoothing operator $H_\iota^{-\infty}(F) \to H^\infty(E)$ by $\mathcal{U}_{-\infty}^*(E,F)$.

If $E = F$, we will just write $\mathcal{U}_{-\infty}^*(E)$.

Remark 2.17. Roe defines in [Roe88a, Definition 5.3] the algebra $\mathcal{U}_{-\infty}(E)$ instead of $\mathcal{U}_{-\infty}^*(E)$, i.e., he does not demand that the adjoint operator is also quasilocal smoothing. The reason why we do this is that we want adjoints of uniform pseudodifferential operators to be again uniform pseudodifferential operators (and the algebra $\mathcal{U}_{-\infty}^*(E)$ is used in the definition of uniform pseudodifferential operators).

2.3 Definition of uniform pseudodifferential operators

Let M^m be an m-dimensional manifold of bounded geometry and let E and F be two vector bundles of bounded geometry over M. Now we will get to the definition of uniform pseudodifferential operators acting on sections of vector bundles of bounded geometry over manifolds of bounded geometry.

Definition 2.18. An operator $P\colon C_c^\infty(E) \to C^\infty(F)$ is a *pseudodifferential operator of order $k \in \mathbb{Z}$*, if with respect to a uniformly locally finite covering $\{B_{2\varepsilon}(x_i)\}$ of M with normal coordinate balls and corresponding subordinate partition of unity $\{\varphi_i\}$ as in Lemma 2.4 we can write

$$P = P_{-\infty} + \sum_i P_i \tag{2.3}$$

[12] Roe calls such operators "*uniform* operators of order k" in [Roe88a, Definition 5.3]. But since the word "uniform" will have another meaning for us (see, e.g., the definition of uniform K-homology), we changed the name.

satisfying the following conditions:

- $P_{-\infty} \in \mathcal{U}^*_{-\infty}(E, F)$, i.e., it is a quasilocal smoothing operator,
- for all i the operator P_i is with respect to synchronous framings of E and F in the ball $B_{2\varepsilon}(x_i)$ a matrix of pseudodifferential operators on $\mathbb{R}^m$ of order k with support[13] in $B_{2\varepsilon}(0) \subset \mathbb{R}^m$, and
- the constants $C_i^{\alpha\beta}$ appearing in the bounds
$$\|D_x^\alpha D_\xi^\beta p_i(x, \xi)\| \leq C_i^{\alpha\beta}(1 + |\xi|)^{k-|\beta|}$$
of the symbols of the operators P_i can be chosen to not depend on i, i.e., there are $C^{\alpha\beta} < \infty$ such that
$$C_i^{\alpha\beta} \leq C^{\alpha\beta} \tag{2.4}$$
for all multi-indices α, β and all i. We will call this the *uniformity condition* for pseudodifferential operators on manifolds of bounded geometry.

We denote the set of all such operators by $\mathrm{U\Psi DO}^k(E, F)$.

From Lemma 2.3 and Lemma 2.5 together with [LM89, Theorem III.§3.12] (and its proof which gives the concrete formula how the symbol of a pseudodifferential operator transforms under a coordinate change) we conclude that the above definition of pseudodifferential operators on manifolds of bounded geometry does neither depend on the chosen uniformly locally finite covering of M by normal coordinate balls, nor on the subordinate partition of unity with uniformly bounded derivatives, nor on the synchronous framings of E and F.

Remark 2.19. We are considering only operators that would correspond to Hörmander's class $S^k_{1,0}(\Omega)$, if we consider open subsets Ω of $\mathbb{R}^m$ instead of an m-dimensional manifold M, i.e., we do not investigate operators corresponding to the more general classes $S^k_{\rho,\delta}(\Omega)$. The paper [Hör67, Definition 2.1] is the one where Hörmander introduced these classes.

Recall that in the case of compact manifolds a pseudodifferential operator P of order k has an extension to a continuous operator $H^s(E) \to H^{s-k}(F)$ for all $s \in \mathbb{Z}$ (see, e.g., [LM89, Theorem III.§3.17(i)]). Due to the uniform local finiteness of the sum in (2.3) and due to the Uniformity Condition (2.4), this result does also hold in our case of a manifold of bounded geometry.

Proposition 2.20. *Let $P \in \mathrm{U\Psi DO}^k(E, F)$. Then P has for all $s \in \mathbb{Z}$ an extension to a continuous operator $P\colon H^s(E) \to H^{s-k}(F)$.*

Remark 2.21. Later we will need the following fact: we can bound the operator norm of $P\colon H^s(E) \to H^{s-k}(F)$ from above by the maximum of the constants $C^{\alpha 0}$ with $|\alpha| \leq K_s$ from the Uniformity Condition (2.4) for P multiplied with a constant C_s, where $K_s \in \mathbb{N}_0$

[13] An operator P is *supported in a subset* K, if $\operatorname{supp} Pu \subset K$ for all u in the domain of P and if $Pu = 0$ whenever we have $\operatorname{supp} u \cap K = \emptyset$.

and C_s only depend on $s \in \mathbb{Z}$ and the dimension of the manifold M. This can be seen by carefully examining the proof of [LM89, Proposition III.§3.2] which is the above proposition for the compact case.[14]

Let us define

$$\mathrm{U\Psi DO}^{-\infty}(E,F) := \bigcap_k \mathrm{U\Psi DO}^k(E,F).$$

We will show $\mathrm{U\Psi DO}^{-\infty}(E,F) = \mathcal{U}^*_{-\infty}(E,F)$: from the previous Proposition 2.20 we conclude that $P \in \mathrm{U\Psi DO}^{-\infty}(E,F)$ is a smoothing operator (using Lemma 2.13). Since we can write $P = P_{-\infty} + \sum_i P_i$, where $P_{-\infty} \in \mathcal{U}^*_{-\infty}(E,F)$ and the P_i are supported in balls with uniformly bounded radii, the operator $\sum_i P_i$ is of finite propagation. So P is the sum of a quasilocal smoothing operator $P_{-\infty}$ and a smoothing operator $\sum_i P_i$ of finite propagation, and therefore a quasilocal smoothing operator. The same arguments also apply to the adjoint P^* of P, so that in the end we can conclude $P \in \mathcal{U}^*_{-\infty}(E,F)$, i.e., we have shown $\mathrm{U\Psi DO}^{-\infty}(E,F) \subset \mathcal{U}^*_{-\infty}(E,F)$.

Since the other inclusion does hold by definition, we get the claim.[15]

Lemma 2.22. $\mathrm{U\Psi DO}^{-\infty}(E,F) = \mathcal{U}^*_{-\infty}(E,F)$.

One of the important properties of pseudodifferential operators on compact manifolds is that the composition of an operator $P \in \mathrm{U\Psi DO}^k(E,F)$ and $Q \in \mathrm{U\Psi DO}^l(F,G)$ is again a pseudodifferential operator of order $k+l$: $PQ \in \mathrm{U\Psi DO}^{k+l}(E,G)$. We can prove this also in our setting by writing

$$\begin{aligned} PQ &= \Big(P_{-\infty} + \sum_i P_i\Big)\Big(Q_{-\infty} + \sum_j Q_j\Big) \\ &= P_{-\infty}Q_{-\infty} + \sum_i P_i Q_{-\infty} + \sum_j P_{-\infty} Q_j + \sum_{i,j} P_i Q_j \end{aligned}$$

and then arguing as follows.

- The first summand is an element of $\mathcal{U}^*_{-\infty}(E,G)$: in [Roe88a, Proposition 5.2] it was shown that the composition of two quasilocal operators is again quasilocal and it is clear that composing smoothing operators again gives smoothing operators, resp. it is easy to see that composing two operators which may be approximated by finite propagation operators again gives such an operator.

- The second and third summands are from $\mathcal{U}^*_{-\infty}(E,G)$ due to Proposition 2.20 and since the sums are uniformly locally finite, the operators P_i and Q_j are supported in coordinate balls of uniform radii (i.e., have finite propagation which is uniformly bounded from above) and their operator norms are uniformly bounded due to the uniformity condition in the definition of pseudodifferential operators.

[14] To be utterly concrete, we have to choose normal coordinate charts and a subordinate partition of unity as in Lemma 2.4 and also synchronous framings for E and F and then use Formula (2.2) which gives Sobolev norms that can be computed locally and that are equivalent to the global norms (2.1).

[15] Of course, our definition of pseudodifferential operators was arranged such that this lemma holds.

- The last summand is a uniformly locally finite sum of pseudodifferential operators of order $k + l$ (here we use the corresponding result for compact manifolds) and to see the Uniformity Condition (2.4) we use [LM89, Theorem III.§3.10]: it states that the symbol of $P_i Q_j$ has formal development $\sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha p_i)(D_x^\alpha q_j)$. So we may deduce the uniformity condition for $P_i Q_j$ from the one for P_i and for Q_j.

Other properties that immediately generalize from the compact to the bounded geometry case is firstly, that the commutator of two pseudodifferential operators is of one order lower than it should a priori be, and secondly, that multiplication with a function $f \in C_b^\infty(M)$ defines a pseudodifferential operator of order 0.

So we have the following important proposition:

Proposition 2.23. $\mathrm{U\Psi DO}^*(E)$ *is a filtered* **-algebra, i.e., for all* $k, l \in \mathbb{Z}$ *we have*

$$\mathrm{U\Psi DO}^k(E) \circ \mathrm{U\Psi DO}^l(E) \subset \mathrm{U\Psi DO}^{k+l}(E),$$

and so $\mathrm{U\Psi DO}^{-\infty}(E)$ *is a two-sided* **-ideal in* $\mathrm{U\Psi DO}^*(E)$.

Furthermore, we have $[\mathrm{U\Psi DO}^k(E), \mathrm{U\Psi DO}^l(E)] \subset \mathrm{U\Psi DO}^{k+l-1}(E)$ *for all* $k, l \in \mathbb{Z}$, *and multiplication with a function* $f \in C_b^\infty(M)$ *defines a pseudodifferential operator of order* 0.

The last property that generalizes to our setting and that we want to mention is the following (the proof of [LM89, Theorem III.§3.9] generalizes directly):

Proposition 2.24. *Let* $P \in \mathrm{U\Psi DO}^k(E, F)$ *be a pseudodifferential operator of arbitrary order and let* $u \in H^s(E)$ *for some* $s \in \mathbb{Z}$.

Then, if u *is smooth on some open subset* $U \subset M$, Pu *is also smooth on* U.

2.4 Uniformity of operators of nonpositive order

Now we get to the important statement that the pseudodifferential operators we have defined are uniform (the discussion here is strongly related to the fact that elliptic uniform pseudodifferential operators will define uniform K-homology classes). Note that we have not yet defined what "uniform" shall mean. This will be done now.

Let $T \in \mathfrak{K}(L^2(E))$ be a compact operator. We know that T is the limit of finite rank operators, i.e., for every $\varepsilon > 0$ there is a finite rank operator k such that $\|T - k\| < \varepsilon$. Now given a collection $\mathcal{A} \subset \mathfrak{K}(L^2(E))$ of compact operators, it may happen that for every $\varepsilon > 0$ the rank needed to approximate an operator from $\mathcal{A}$ may be bounded from above by a common bound for all operators. This is formalized in the following definition.

Definition 2.25 (Uniformly approximable collections of operators)**.** A collection of operators $\mathcal{A} \subset \mathfrak{K}(L^2(E))$ is said to be *uniformly approximable*, if for every $\varepsilon > 0$ there is an $N > 0$ such that for every $T \in \mathcal{A}$ there is a rank-N operator k with $\|T - k\| < \varepsilon$.

Examples 2.26. Every collection of finite rank operators with uniformly bounded rank is uniformly approximable.

Furthermore, every finite collection of compact operators is uniformly approximable and so also every totally bounded subset of $\mathfrak{K}(L^2(E))$.

The converse is in general false since a uniformly approximable family need not be bounded (take infinitely many rank-1 operators with operator norms going to infinity).

Even if we assume that the uniformly approximable family is bounded we do not necessarily get a totally bounded set: let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(E)$ and P_i the orthogonal projection onto the 1-dimensional subspace spanned by the vector e_i. Then the collection $\{P_i\} \subset \mathfrak{K}(L^2(E))$ is uniformly approximable (since all operators are of rank 1) but not totally bounded (since $\|P_i - P_j\| = 1$ for $i \neq j$)[16].

Let us define

$$L\text{-}\operatorname{Lip}_R(M) := \{f \in C_c(M) \mid f \text{ is } L\text{-Lipschitz}, \operatorname{diam}(\operatorname{supp} f) \leq R \text{ and } \|f\|_\infty \leq 1\}.$$

Definition 2.27 ([Špa09, Definition 2.3])**.** Let $T \in \mathfrak{B}(L^2(E))$. We say that T is *uniformly locally compact*, if for every $R, L > 0$ the collection

$$\{fT, Tf \mid f \in L\text{-}\operatorname{Lip}_R(M)\}$$

is uniformly approximable.

We say that T is *uniformly pseudolocal*, if for every $R, L > 0$ the collection

$$\{[T, f] \mid f \in L\text{-}\operatorname{Lip}_R(M)\}$$

is uniformly approximable.

Remark 2.28. In [Špa09] uniformly locally compact operators were called "*l*-uniform" and uniformly pseudolocal operators "*l*-uniformly pseudolocal".

We will now show that pseudodifferential operators of negative order are uniformly locally compact and that pseudodifferential operators of order 0 are uniformly pseudolocal. We will start with the operators of negative order.

Proposition 2.29. *Let $A \in \mathfrak{B}(L^2(E))$ be a finite propagation operator of negative order $k < 0$[17] such that its adjoint A^* also has finite propagation and is of negative order $k' < 0$. Then A is uniformly locally compact. Even more, the collection*

$$\{fT, Tf \mid f \in B_R(M)\}$$

is uniformly approximable for all $R, L > 0$, where $B_R(M)$ consists of all bounded Borel functions h on M with $\operatorname{diam}(\operatorname{supp} h) \leq R$ *and* $\|h\|_\infty \leq 1$.

[16] Another way to see that $\{P_i\}$ is not totally bounded is to use the characterization of totally bounded subsets of $\mathfrak{K}(H)$ from [AP68, Theorem 3.5]: a family $\mathcal{A} \subset \mathfrak{K}(H)$ is totally bounded if and only if both $\mathcal{A}$ and $\mathcal{A}^*$ are collectively compact, i.e., the sets $\{Tv \mid T \in \mathcal{A}, v \in H \text{ with } \|v\| = 1\} \subset H$ and $\{T^*v \mid T \in \mathcal{A}, v \in H \text{ with } \|v\| = 1\} \subset H$ have compact closure.

[17] See Definition 2.15. Note that we do not assume that A is a pseudodifferential operator.

Proof. Let $f \in B_R(M)$, $K := \operatorname{supp} f \subset M$ and r be the propagation of A. The operator $\chi A f = Af$, where χ is the characteristic function of $B_r(K)$, factores as

$$L^2(E) \xrightarrow{\cdot f} L^2(E|_K) \xrightarrow{\chi \cdot A} H^{-k}(E|_{B_r(K)}) \hookrightarrow L^2(E|_{B_r(K)}) \to L^2(E).$$

The following properties hold:

- multiplication with f has operator norm ≤ 1, since $\|f\|_\infty \leq 1$, and analogously for the multiplication with χ,
- the norm of $\chi \cdot A\colon L^2(E|_K) \to H^{-k}(E|_{B_r(K)})$ can be bounded from above by the norm of $A\colon L^2(E) \to H^{-k}(E)$ (i.e., the upper bound does not depend on K nor r),
- the inclusion $H^{-k}(E|_{B_r(K)}) \hookrightarrow L^2(E|_{B_r(K)})$ is compact (due to the Theorem of Rellich–Kondrachov) and this compactness is uniform, i.e., its approximability by finite rank operators[18] depends only on R (the upper bound for the diameter of $\operatorname{supp} f$) and r, but not on K (this uniformity is due to the bounded geometry of M and of the bundles E and F), and
- the inclusion $L^2(E|_{B_r(K)}) \to L^2(E)$ is of norm ≤ 1.

From this we conclude that the operator $\chi A f = Af$ is compact and this compactness is uniform, i.e., its approximability by finite rank operators depends only on R and r. So we can conclude that $\{Af \mid f \in B_R(M)\}$ is uniformly approximable.

Applying the same reasoning to the adjoint operator,[19] we conclude that A is uniformly locally compact. □

Using an approximation argument[20] we may also show the following corollary:

Corollary 2.30. *Let A be a quasilocal operator of negative order and let the same hold true for its adjoint A^*. Then A is uniformly locally compact; in fact, it even satisfies the stronger condition from the above Proposition 2.29.*

Proof. We have to show that $\{Af \mid f \in B_R(M)\}$ is uniformly approximable. Let $\varepsilon > 0$ be given and let r_ε be such that $\mu_A(r) < \varepsilon$ for all $r \geq r_\varepsilon$, where μ_A is the dominating function of A. Then $\chi_{B_{r_\varepsilon}(\operatorname{supp} f)} Af$ is ε-away from Af and the same reasoning as in the proof of the above Proposition 2.29 shows that the approximability (up to an error of ε) of $\chi_{B_{r_\varepsilon}(\operatorname{supp} f)} Af$ does only depend on R and r_ε. From this the claim that $\{Af \mid f \in B_R(M)\}$ is uniformly approximable follows.

Using the adjoint operator and the same arguments for it, we conclude that A is uniformly locally compact. □

[18]Here we mean the existence of an upper bound on the rank needed to approximate the operator by finite rank operators, given an $\varepsilon > 0$.

[19]By assumption the adjoint operator also has finite propagation and is of negative order. So we conclude that $\{A^* f \mid f \in B_R(M)\}$ is uniformly approximable. But a collection $\mathcal{A}$ of compact operators is uniformly approximable if and only if the adjoint family $\mathcal{A}^*$ is uniformly approximable. So we get that $\{(A^* f)^* = \overline{f} A \mid f \in B_R(M)\}$ is uniformly approximable.

[20]Note that we will not approximate the quasilocal operator A itself by finite propagation operators in this argument. In fact, it is an open problem whether quasilocal operators may be approximated by finite propagation operators.

Corollary 2.31. *Let $P \in \mathrm{U\Psi DO}^k(E)$ be a pseudodifferential operator of negative order $k < 0$. Then P is uniformly locally compact.*

Let us now get to the case of pseudodifferential operators of order 0, where we want to show that such operators are uniformly pseudolocal.

Recall the following fact for compact manifolds: T is pseudolocal[21] if and only if fTg is a compact operator for all $f, g \in C(M)$ with disjoint supports. This observation is due to Kasparov and a proof might be found in, e.g., [HR00, Proposition 5.4.7]. We can add another equivalent characterization which is basically also proved in the cited proposition: an operator T is pseudolocal if and only if fTg is a compact operator for all bounded Borel functions f and g on M with disjoint supports.

We have analogous equivalent characterizations for uniformly pseudolocal operators, which we will state in the following lemma. The proof of it is similar to the compact case (and uses the fact that the subset of all uniformly pseudolocal operators is closed in operator norm, which is proved in [Špa09, Lemma 4.2]). Furthermore, in order to prove that the Points 4 and 5 in the statement of the next lemma are equivalent to the other points we need the bounded geometry of M. For the convenience of the reader we will give a full proof of the lemma.

Let us introduce the notions $B_b(M)$ for all bounded Borel functions on M and $B_R(M)$ for its subset consisting of all function h with $\operatorname{diam}(\operatorname{supp} h) \le R$ and $\|h\|_\infty \le 1$.

Lemma 2.32. *The following are equivalent for an operator $T \in \mathfrak{B}(L^2(E))$:*

1. *T is uniformly pseudolocal,*
2. *for all $R, L > 0$ the following collection is uniformly approximable:*
$$\{fTg, gTf \mid f \in B_b(M),\ \|f\|_\infty \le 1,\ g \in L\text{-}\mathrm{Lip}_R(M),\ \operatorname{supp} f \cap \operatorname{supp} g = \emptyset\},$$
3. *for all $R, L > 0$ the following collection is uniformly approximable:*
$$\{fTg, gTf \mid f \in B_b(M),\ \|f\|_\infty \le 1,\ g \in B_R(M),\ d(\operatorname{supp} f, \operatorname{supp} g) \ge L\},$$
4. *for every $L > 0$ there is a sequence $(L_j)_{j \in \mathbb{N}}$ of positive numbers (not depending on the operator T) such that*
$$\begin{aligned}\{fTg, gTf \mid\ & f \in B_b(M) \text{ with } \|f\|_\infty \le 1,\\ & g \in B_R(M) \cap C_b^\infty(M) \text{ with } \|\nabla^j g\|_\infty \le L_j, \text{ and}\\ & \operatorname{supp} f \cap \operatorname{supp} g = \emptyset\}\end{aligned}$$
is uniformly approximable for all $R, L > 0$.
5. *for every $L > 0$ there is a sequence $(L_j)_{j \in \mathbb{N}}$ of positive numbers (not depending on the operator T) such that*
$$\{[T, g] \mid g \in B_R(M) \cap C_b^\infty(M) \text{ with } \|\nabla^j g\|_\infty \le L_j\}$$
is uniformly approximable for all $R, L > 0$.

[21] That is to say, $[T, f]$ is a compact operator for all $f \in C_c(M)$.

Proof. **1 ⇒ 2:** Let $f \in B_b(M)$ with $\|f\|_\infty \leq 1$ and $g \in L\text{-}\operatorname{Lip}_R(M)$ have disjoint supports, i.e., $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. From the latter we conclude $fTg = f[T,g]$, from which the claim follows (because T is uniformly pseudolocal and because the operator norm of multiplication with f is ≤ 1). Of course such an argument also works with the roles of f and g changed.

2 ⇒ 3: Let $f \in B_b(M)$ with $\|f\|_\infty \leq 1$ and $g \in B_R(M)$ with $d(\operatorname{supp} f, \operatorname{supp} g) \geq L$. We define $g'(x) := \max\big(0, 1 - 1/L \cdot d(x, \operatorname{supp} g)\big) \in 1/L\text{-}\operatorname{Lip}_{R+2L}(M)$. Since $g'g = g$, the claim follows from writing $fTg = fTg'g$ and because multiplication with g has operator norm ≤ 1, and we of course also may change the roles of f and g.

3 ⇒ 1: Let $f \in L\text{-}\operatorname{Lip}_R(M)$. For given $\varepsilon > 0$ we partition the range of f into a sequence of non-overlapping half-open intervals $U_1, \ldots, U_n$, each having diameter less than ε, such that $\overline{U_i}$ intersects $\overline{U_j}$ if and only if $|i-j| \leq 1$. Denoting by χ_i the characteristic function of $f^{-1}(U_i)$, we get that $\chi_i \in B_R(M)$ if $0 \notin U_i$, since the support of f has diameter less than or equal to R, and furthermore $d(\operatorname{supp}\chi_i, \operatorname{supp}\chi_j) \geq \frac{\varepsilon}{L}$ if $|i-j| > 1$, since f is L-Lipschitz.

By Point 3 we have that the collections $\{\chi_i T \chi_j, \chi_j T \chi_i\}$ are uniformly approximable for all i,j with $|i-j| > 1$. Choosing points $x_1, \ldots, x_n$ from $f^{-1}(U_1), \ldots, f^{-1}(U_n)$ and defining $f' := f(x_1)\chi_1 + \cdots + f(x_n)\chi_n$, we get $\|f - f'\|_\infty < \varepsilon$. The operator $[T,f]$ is $2\varepsilon\|T\|$-away from $[T,f']$, and since $\chi_1 + \cdots + \chi_n = 1$ we have

$$Tf' - f'T = \sum_{i,j} \chi_j T f(x_i)\chi_i - f(x_j)\chi_j T \chi_i.$$

Since we already know that $\{\chi_i T \chi_j, \chi_j T \chi_i\}$ are uniformly approximable for all i,j with $|i-j| > 1$, it remains to treat the sum (note that the summand for $i = j$ is zero)

$$\sum_{|i-j|=1} \chi_j T f(x_i)\chi_i - f(x_j)\chi_j T \chi_i = \sum_{|i-j|=1} \big(f(x_i) - f(x_j)\big)\chi_j T \chi_i.$$

We split the sum into two parts, one where $i = j+1$ and the other one where $i = j-1$. The first part takes the form

$$\sum_j \big(f(x_{j+1}) - f(x_j)\big)\chi_j T \chi_{j+1},$$

i.e., is a direct sum of operators from $\chi_{j+1} \cdot L^2(E)$ to $\chi_j \cdot L^2(E)$. Therefore its norm is the maximum of the norms of its summands. But the latter are $\leq 2\varepsilon\|T\|$ since $|f(x_{j+1}) - f(x_j)| \leq 2\varepsilon$. We treat the second part of the sum in the above display the same way and conclude that the sum in the above display is in norm $\leq 4\varepsilon T$. Putting it all together it follows that T is the operator norm limit of uniformly pseudolocal operators, from which it follows that T itself is uniformly pseudolocal (it is proved in [Špa09, Lemma 4.2] that the uniformly pseudolocal operators are closed in operator norm, as are also the uniformly locally compact ones).

2 ⇒ 4: Clear. We have to set $L_1 := L$ and the other values $L_{j\geq 2}$ do not matter (i.e., may be set to something arbitrary).

4 ⇒ 3: This is similar to the proof of 2 ⇒ 3, but we have to smooth the function g' constructed there. Let us make this concrete, i.e., let $f \in B_b(M)$ with $\|f\|_\infty \le 1$ and $g \in B_R(M)$ with $d(\operatorname{supp} f, \operatorname{supp} g) \ge L$ be given. We define

$$g'(x) := \max\big(0, 1 - 2/L \cdot d(x, B_{L/4}(\operatorname{supp} g))\big) \in 2/L\text{-}\operatorname{Lip}_{R+3L/2}(M).$$

Note that $g' \equiv 1$ on $B_{L/4}(\operatorname{supp} g)$ and $g' \equiv 0$ outside $B_{3L/4}(\operatorname{supp} g)$. We cover M by normal coordinate charts and choose a "nice" subordinate partition of unity φ_i as in Lemma 2.4. If ψ is now a mollifier on $\mathbb{R}^m$ supported in $B_{L/8}(0)$, we apply it in every normal coordinate chart to $\varphi_i g'$ and reassemble then all the mollified parts of g' again to a (now smooth) function g'' on M. This function g'' is now supported in $B_{7L/8}(\operatorname{supp} g)$, and is constantly 1 on $B_{L/8}(\operatorname{supp} g)$. So $fTg = fTg''g$ from which we may conclude the uniform approximability of the collection $\{fTg\}$ for f and g satisfying $f \in B_b(M)$ with $\|f\|_\infty \le 1$ and $g \in B_R(M)$ with $d(\operatorname{supp} f, \operatorname{supp} g) \ge L$. Note that the constants L_j appearing in $\|\nabla^j g''\|_\infty \le L_j$ depend on L, φ_i and ψ, but not on f, g or R. The dependence on φ_i and ψ is ok, since we may just fix a particular choice of them (note that the choice of ψ also depends on L), and the dependence on L is explicitly stated in the claim.

Of course we may also change the roles of f and g in this argument.

5 ⇒ 4: Clear. We just have to write $fTg = f[T, g]$ and analogously for gTf.

1 ⇒ 5: Clear. □

With the above lemma at our disposal we may now prove the following proposition.

Proposition 2.33. *Let $P \in \mathrm{U\Psi DO}^0(E)$. Then P is uniformly pseudolocal.*

Proof. Writing $P = P_{-\infty} + \sum_i P_i$ with $P_{-\infty} \in \mathcal{U}^*_{-\infty}(E)$, we may without loss of generality assume that P has finite propagation R' (since $P_{-\infty}$ is uniformly locally compact by the above Corollary 2.30 and uniformly locally compact operators are uniformly pseudolocal).

We will use the equivalent characterization in Point 4 of the above lemma: let $R, L > 0$ and the corresponding sequence $(L_j)_{j \in \mathbb{N}}$ be given. We have to show that

$$\begin{aligned}\{fPg, gPf \mid f \in B_b(M) &\text{ with } \|f\|_\infty \le 1,\\ &g \in B_R(M) \cap C_b^\infty(M) \text{ with } \|\nabla^j g\|_\infty \le L_j, \text{ and}\\ &\operatorname{supp} f \cap \operatorname{supp} g = \emptyset\}\end{aligned}$$

is uniformly approximable for all $R, L > 0$.

We have

$$fPg = f\chi_{B_{R'}(\operatorname{supp} g)} Pg = f\chi_{B_{R'}(\operatorname{supp} g)}[P, g]$$

since the supports of f and g are disjoint.

With Proposition 2.23 we conclude that multiplication with g is a pseudodifferential operator of order 0 (since $g \in C_b^\infty(M)$) and furthermore, that the commutator $[P, g]$ is a pseudodifferential operator of order -1. Therefore, by the above Corollary 2.31, we know that the set $\{f\chi_{B_{R'}(\operatorname{supp} g)}[P, g] \mid f \in B_R(M)\}$ is uniformly approximable. So we conclude that our operators $f[P, g]$ have the needed uniformity in the functions f.

It remains to show that we also have the needed uniformity in the functions g. Writing $P = \sum_i P_i$[22], we get $[P, g] = \sum_i [P_i, g]$. Now each $[P_i, g]$ is a pseudodifferential operator of order -1, their supports[23] depend only on the propagation of P and on the value of R (but not on i nor on the concrete choice of g) and their operator norms as maps $L^2(E) \to H^1(E)$ are bounded from above by a constant that only depends on P, on R and on the values of all the L_j (but again, neither on i nor on g). The last fact follows from a combination of Remark 2.21 together with the estimates on the symbols of the $[P_i, g]$ that we get from the proof that they are pseudodifferential operators of order -1. So examining the proof of Proposition 2.29 more closely, we see that these properties suffice to conclude the needed uniformity of $f[P, g]$ in the functions g.

The operators gPf may be treated analogously. □

2.5 Elliptic operators

In this section we will define the notion of ellipticity for uniform pseudodifferential operators and important consequences of it (elliptic regularity, fundamental elliptic estimates and essential self-adjointness). Let π^*E and π^*F denote the pull-back bundles of E and F to the cotangent bundle $\pi\colon T^*M \to M$ of the m-dimensional manifold M.

Definition 2.34 (Symbols)**.** Let p be a section of the bundle $\operatorname{Hom}(\pi^*E, \pi^*F)$ over T^*M. We call p a *symbol of order* $k \in \mathbb{Z}$, if the following holds: choosing a uniformly locally finite covering $\{B_{2\varepsilon}(x_i)\}$ of M through normal coordinate balls and corresponding subordinate partition of unity $\{\varphi_i\}$ as in Lemma 2.4, and choosing synchronous framings of E and F in these balls $B_{2\varepsilon}(x_i)$, we can write p as a uniformly locally finite sum $p = \sum_i p_i$, where $p_i(x, \xi) := p(x, \xi)\varphi(x)$ for $x \in M$ and $\xi \in T^*_x M$, and interpret each p_i as a matrix-valued function on $B_{2\varepsilon}(x_i) \times \mathbb{C}^m$. Then for all multi-indices α and β there must exist a constant $C^{\alpha\beta} < \infty$ such that for all i and all x, ξ we have

$$\|D_x^\alpha D_\xi^\beta p_i(x, \xi)\| \le C^{\alpha\beta}(1 + |\xi|)^{k - |\beta|}. \tag{2.5}$$

We denote the vector space all symbols of order $k \in \mathbb{Z}$ by $\operatorname{Symb}^k(E, F)$.

From Lemma 2.3 and Lemma 2.5 we conclude that the above definition of symbols does neither depend on the chosen uniformly locally finite covering of M through normal coordinate balls, nor on the subordinate partition of unity (as long as the functions $\{\varphi_i\}$ have uniformly bounded derivatives), nor on the synchronous framings of E and F.

If all the choices above are fixed, we immediately see from the definition of pseudodifferential operators that an operator $P \in \mathrm{U\Psi DO}^k(E, F)$ has a symbol $p \in \operatorname{Symb}^k(E, F)$. Analogously as in the case of compact manifolds,[24] we may show that if we make other choices for the coordinate charts, subordinate partition of unity and synchronous framings, the symbol p of P changes by an element of $\operatorname{Symb}^{k-1}(E, F)$. So P has a well-defined principal symbol class $[p] \in \operatorname{Symb}^k(E, F) / \operatorname{Symb}^{k-1}(E, F) =: \operatorname{Symb}^{k-[1]}(E, F)$.

[22]Recall that we assumed without loss of generality that there is no $P_{-\infty}$.

[23]Recall that an operator P is *supported in a subset* K, if $\operatorname{supp} Pu \subset K$ for all u in the domain of P and if $Pu = 0$ whenever we have $\operatorname{supp} u \cap K = \emptyset$.

[24]see, e.g., [LM89, Theorem III.§3.19]

Definition 2.35 (Elliptic symbols)**.** Let $p \in \mathrm{Symb}^k(E, F)$. Recall that p is a section of the bundle $\mathrm{Hom}(\pi^* E, \pi^* F)$ over $T^* M$. We will call p *elliptic*, if there is an $R > 0$ such that $p|_{|\xi|>R}$[25] is invertible and this inverse p^{-1} satisfies the Inequality (2.5) for $\alpha, \beta = 0$ and order $-k$ (and of course only for $|\xi| > R$ since only there the inverse is defined). Note that as in the compact case it follows that p^{-1} satisfies the Inequality (2.5) for all multi-indices α, β.

The proof of the following lemma is easy.

Lemma 2.36. *If $p \in \mathrm{Symb}^k(E, F)$ is elliptic, then every other representative p' of the class $[p] \in \mathrm{Symb}^{k-[1]}(E, F)$ is also elliptic.*

Due to the above lemma we are now able to define what it means for a pseudodifferential operator to be elliptic:

Definition 2.37 (Elliptic UΨDOs)**.** Let $P \in \mathrm{U\Psi DO}^k(E, F)$. We will call P *elliptic*, if its principal symbol $\sigma(P)$ is elliptic.

The importance of elliptic operators lies in the fact that they admit an inverse modulo operators of order $-\infty$. We may prove this analogously as in the case of a compact manifold. See also [Kor91, Theorem 3.3] where Kordyukov proves the existence of parametrices for his class of pseudodifferential operators (which is our class restricted to operators of finite propagation).

Theorem 2.38 (Existence of parametrices)**.** *Let $P \in \mathrm{U\Psi DO}^k(E, F)$ be elliptic.*

Then there exists an operator $Q \in \mathrm{U\Psi DO}^{-k}(F, E)$ such that

$$PQ = \mathrm{id} - S_1 \text{ and } QP = \mathrm{id} - S_2,$$

where $S_1 \in \mathrm{U\Psi DO}^{-\infty}(F)$ and $S_2 \in \mathrm{U\Psi DO}^{-\infty}(E)$.

Using parametrices, we can prove a lot of the important properties of elliptic operators, e.g., *elliptic regularity* (which is a converse to Proposition 2.24 and a proof of it may be found in, e.g. [LM89, Theorem III.§4.5]):

Theorem 2.39 (Elliptic regularity)**.** *Let $P \in \mathrm{U\Psi DO}^k(E, F)$ be elliptic and let furthermore $u \in H^s(E)$ for some $s \in \mathbb{Z}$.*

Then, if Pu is smooth on an open subset $U \subset M$, u is already smooth on U. Furthermore, for $k > 0$: if $Pu = \lambda u$ on U for some $\lambda \in \mathbb{C}$, then u is smooth on U.

Later we will also need the following *fundamental elliptic estimate* (the proof from [LM89, Theorem III.§5.2(iii)] generalizes directly):

Theorem 2.40 (Fundamental elliptic estimate)**.** *Let $P \in \mathrm{U\Psi DO}^k(E, F)$ be elliptic. Then for each $s \in \mathbb{Z}$ there is a constant $C_s > 0$ such that*

$$\|u\|_{H^s(E)} \le C_s \big(\|u\|_{H^{s-k}(E)} + \|Pu\|_{H^{s-k}(F)} \big)$$

for all $u \in H^s(E)$.

[25]This notation means the following: we restrict p to the bundle $\mathrm{Hom}(\pi^* E, \pi^* F)$ over the space $\{(x, \xi) \in T^* M \mid |\xi| > R\} \subset T^* M$.

Another important implication of ellipticity is that symmetric[26], elliptic pseudodifferential operators of positive order are essentially self-adjoint[27]. We need this since we will have to consider functions of pseudodifferential operators. But first we will show that a symmetric and elliptic operator is also symmetric as an operator on Sobolev spaces.

Lemma 2.41. *Let $P \in \mathrm{U}\Psi\mathrm{DO}^k(E)$ with $k \geq 1$ be symmetric on $L^2(E)$ and elliptic. Then P is also symmetric on the Sobolev spaces $H^{lk}(E)$ for $l \in \mathbb{Z}$, where we use on $H^{lk}(E)$ the scalar product as described in the proof.*

Proof. Due to the fundamental elliptic estimate the norm $\|u\|_{H^0} + \|Pu\|_{H^0}$ (note that $H^0(E) = L^2(E)$ by definition) on $H^k(E)$ is equivalent to the usual[28] norm $\|u\|_{H^k}$ on it. Now $\|u\|_{H^0} + \|Pu\|_{H^0}$ is equivalent to $\big(\|u\|^2_{H^0} + \|Pu\|^2_{H^0}\big)^{1/2}$ which is induced by the scalar product

$$\langle u, v\rangle_{H^k,P} := \langle u, v\rangle_{H^0} + \langle Pu, Pv\rangle_{H^0}.$$

Since P is symmetric for the H^0-scalar product, we immediately see that it is also symmetric for this particular scalar product $\langle \cdot, \cdot\rangle_{H^k,P}$ on $H^k(E)$.

To extend to the Sobolev spaces $H^{lk}(E)$ for $l > 0$ we repeatedly invoke the above arguments, e.g., on $H^{2k}(E)$ we have the equivalent norm $\big(\|u\|^2_{H^k,P} + \|Pu\|^2_{H^k,P}\big)^{1/2}$ (again due to the fundamental elliptic estimate) which is induced by the scalar product $\langle u, v\rangle_{H^k,P} + \langle Pu, Pv\rangle_{H^k,P}$ and now we may use that we already know that P is symmetric with respect to $\langle \cdot, \cdot\rangle_{H^k,P}$.

Finally, for $H^{lk}(E)$ for $l < 0$ we use the fact that they are the dual spaces to $H^{-lk}(E)$ where we know that P is symmetric, i.e., we equip $H^{lk}(E)$ for $l < 0$ with the scalar product induced from the duality: $\langle u, v\rangle_{H^{lk},P} := \langle u', v'\rangle_{H^{-lk},P}$, where $u', v' \in H^{-lk}(E)$ are the dual vectors to $u, v \in H^{lk}(E)$ (note that the induced norm on $H^{lk}(E)$ is exactly the operator norm if we regard $H^{lk}(E)$ as the dual space of $H^{-lk}(E)$). $\square$

Now we get to the proof that elliptic and symmetric operators are essentially self-adjoint. Note that if we work with differential operators D of first order on open manifolds we do not need ellipticity for this result to hold, but weaker conditions suffice, e.g., that the symbol σ_D of D satisfies $\sup_{x \in M, \|\xi\|=1} \|\sigma_D(x, \xi)\| < \infty$ (by the way, this condition is incorporated in our definition of pseudodifferential operators by the uniformity condition). But if we want essential self-adjointness of higher order operators, we have to assume stronger conditions (see the counterexample [Tau10]).

Proposition 2.42 (Essential self-adjointness)**.** *Let $P \in \mathrm{U}\Psi\mathrm{DO}^k(E)$ with $k \geq 1$ be elliptic and symmetrid. Then the unbounded operator $P\colon H^{lk}(E) \to H^{lk}(E)$ is essentially self-adjoint for all $l \in \mathbb{Z}$, where we equip these Sobolev spaces with the scalar products as described in the proof of the above Lemma 2.41.*

[26]This means that we have $\langle Pu, v\rangle_{L^2(E)} = \langle u, Pv\rangle_{L^2(E)}$ for all $u, v \in C_c^\infty(E)$.

[27]Recall that a symmetric, unbounded operator is called *essentially self-adjoint*, if its closure is a self-adjoint operator.

[28]We have of course possible choices here, e.g., the global norm (2.1) or the local definition (2.2), but they are all equivalent to each other since M and E have bounded geometry.

Proof. This proof is an adapted version of the proof of this statement for compact manifolds from [Tau10].

We will use the following sufficient condition for essential self-adjointness: if we have a symmetric and densely defined operator T such that $\ker(T^* \pm i) = \{0\}$, then the closure $\overline{T}$ of T is self-adjoint and is the unique self-adjoint extension of T.

So let $u \in \ker(P^* \pm i) \subset H^{lk}(E)$, i.e., $P^* u = \pm iu$. From elliptic regularity we get that u is smooth and using the fundamental elliptic estimate for P^*[29] we can then conclude $\|u\|_{H^{k+lk}} \leq C_{k+lk}\big(\|u\|_{H^{lk}} + \|P^* u\|_{H^{lk}}\big) = 2C_{k+lk}\|u\|_{H^{lk}} < \infty$, i.e., $u \in H^{k+lk}(E)$. Repeating this argument gives us $u \in H^\infty(E)$, i.e., u lies in the domain of P itself and is therefore an eigenvector of it to the eigenvalue $\pm i$. But since P is symmetric we must have $u = 0$. This shows $\ker(P^* \pm i) = \{0\}$ and therefore P is essentially self-adjoint. □

2.6 Functions of elliptic, symmetric operators

Let $P \in \mathrm{U\Psi DO}^k(E)$ be an elliptic and symmetric pseudodifferential operator of positive order $k \geq 1$. By Proposition 2.42 we know that $P\colon L^2(E) \to L^2(E)$ is essentially self-adjoint. So, if f is a Borel function defined on the spectrum of P, the operator $f(P)$ is defined by the functional calculus. In this whole section P will denote such an operator, i.e., a symmetric and elliptic one of positive order.

Given such a pseudodifferential operator P, we will later show that it defines naturally a class in uniform K-homology. For this we will have to consider $\chi(P)$, where χ is a so-called normalizing function, and we will have to show that $\chi(P)$ is uniformly pseudolocal and $\chi(P)^2 - 1$ is uniformly locally compact. For this we will need the analysis done in this section, i.e., this section is purely technical in nature.

If f is a Schwartz function, we have the formula $f(P) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) e^{itP} dt$, where $\hat{f}$ is the Fourier transform of f. In the case that $P = D$ is an elliptic, first-order differential operator and its symbol satisfies $\sup_{x \in M, \|\xi\|=1} \|\sigma_D(x,\xi)\| < \infty$, the operator e^{itD} has finite propagation (a proof of this may be found in, e.g., [HR00, Proposition 10.3.1]) from which (exploiting the above formula for $f(D)$) we may deduce the needed properties of $\chi(P)$ and $\chi(P)^2 - 1$. But this is no longer the case for a general elliptic pseudodifferential operator P and therefore the analysis that we have to do here in this general case is much more sophisticated.

Note that the restriction to operators of order $k \geq 1$ in this section is no restriction on the fact that elliptic pseudodifferential operators define uniform K-homology classes. In fact, if P has order $k \leq 0$, then we already know from Proposition 2.33 that P is uniformly pseudolocal, i.e., there is no need to form the expression $\chi(P)$ in order for P to define a uniform K-homology class.

We start with the following crucial technical lemma which is a generalization of the fact that e^{itD} has finite propagation to pseudodifferential operators. Note that we do not have to assume something like $\sup_{x \in M, \|\xi\|=1} \|\sigma_D(x,\xi)\| < \infty$ that we had to for first-order differential operators, since such an assumption is subsumed in the uniformity condition that we have in the definition of pseudodifferential operators.

[29] Note that P^* is elliptic if and only if P is.

Lemma 2.43. *Let $P \in \mathrm{U\Psi DO}^{k\geq 1}(E)$ be symmetric and elliptic. Then the operator e^{itP} is a quasilocal operator $H^{lk}(E) \to H^{lk-k}(E)$ for all $l \in \mathbb{Z}$ and $t \in \mathbb{R}$.*

Proof. This proof is a watered down version of the proof of [MM13, Theorem 3.1].

We will need the following two facts:

1. $\|e^{itP}\|_{lk,lk} = 1$ for all $r \in \mathbb{Z}$, where $\|\cdot\|_{lk,lk}$ denotes the operator norm of operators $H^{lk}(E) \to H^{lk}(E)$ and
2. there is a $\kappa > 0$ such that $\|[\eta, P]\|_{s,s-(k-1)} \leq \kappa \cdot \sum_{j=1}^{N} \|\nabla^j \eta\|_\infty$ for all smooth $\eta \in C_b^\infty(M)$, where N depends on $s \in \mathbb{Z}$ and the dimension of M.

The first one holds since e^{itP} is a unitary operator (using Proposition 2.42) and the second is due to the facts that by Proposition 2.23 the commutator $[\eta, P]$ is a pseudodifferential operator of order $k-1$ (recall that smooth functions with bounded derivatives are operators of order 0) and due to Remark 2.21 (where we have to recall the formula how to compute the symbol of the composition of two pseudodifferential operators from, e.g., [LM89, Theorem III.§3.10]).

Let $L \subset M$ and let $u \in H^{lk}(E)$ be supported within L. Furthermore, we choose an $R > 0$ and a smooth, real-valued function η with $\eta \equiv 1$ on a neighbourhood of $\operatorname{supp} u$, $\eta \equiv 0$ on $M - B_{R+1}(L)$ and the first N derivatives of η bounded from above by C/R for a fixed constant C. Then we have for all $v \in H^{lk-k}(E)$ that are supported in $M - B_{R+1}(L)$

$$\begin{aligned}\langle e^{itP}u, v\rangle_{H^{lk-k}} &= \langle e^{itP}\eta u, v\rangle_{H^{lk-k}} - \langle e^{itP}u, \eta v\rangle_{H^{lk-k}}\\ &= \langle [e^{itP}, \eta]u, v\rangle_{H^{lk-k}},\end{aligned}$$

i.e., $|\langle e^{itP}u, v\rangle_{H^{lk-k}}| \leq \|[e^{itP}, \eta]\|_{lk,lk-k} \cdot \|u\|_{H^{lk}} \cdot \|v\|_{H^{lk-k}}$ and it remains to give an estimate for $\|[e^{itP}, \eta]\|_{lk,lk-k}$: We have

$$\begin{aligned}[e^{itP}, \eta] &= \int_0^1 \tfrac{d}{dx}\big(e^{ixtP}\eta e^{i(1-x)tP}\big)dx\\ &= -it\int_0^1 e^{ixtP}[\eta, P]e^{i(1-x)tP}dx\end{aligned}$$

which gives by factorizing the integrand as

$$H^{lk}(E) \stackrel{e^{i(1-x)tP}}{\longrightarrow} H^{lk}(E) \stackrel{[\eta,P]}{\longrightarrow} H^{lk-(k-1)}(E) \hookrightarrow H^{lk-k}(E) \stackrel{e^{ixtP}}{\longrightarrow} H^{lk-k}(E)$$

the estimate

$$\|[e^{itP}, \eta]\|_{lk,lk-k} \leq |t| \int_0^1 \|[\eta, P]\|_{lk,lk-(k-1)}dx \leq |t| \cdot \kappa \cdot \sum_{j=1}^{N} \|\nabla^j \eta\|_\infty.$$

Since $\|\nabla^j \eta\|_\infty < C/R$ for all $1 \leq j \leq N$, we have shown

$$|\langle e^{itP}u, v\rangle_{H^{lk-k}}| < \frac{|t|\kappa NC}{R} \cdot \|u\|_{H^{lk}} \cdot \|v\|_{H^{lk-k}} \tag{2.6}$$

for all u supported in L and all v in $M - B_{R+1}(L)$. Because $R > 0$ and $l \in \mathbb{Z}$, $t \in \mathbb{R}$ were arbitrary, the claim that e^{itP} is a quasilocal operator $H^{lk}(E) \to H^{lk-k}(E)$ for all $l \in \mathbb{Z}$ and $t \in \mathbb{R}$ follows. ∎

Corollary 2.44 (cf. [Tay81, Lemma 1.1 in Chapter XII.§1])**.** *Let $q(t)$ be a function on $\mathbb{R}$ such that for an $n \in \mathbb{N}_0$ the functions $q(t)|t|$, $q'(t)|t|$, ..., $q^{(n)}(t)|t|$ are integrable, i.e., belong to $L^1(\mathbb{R})$.*

Then the operator defined by $\int_{\mathbb{R}} q(t)e^{itP}dt$ is for all $l \in \mathbb{Z}$ a quasilocal operator $H^{lk-nk+k}(E) \to H^{lk}(E)$, i.e., is of order $-nk+k$.

Proof. Let $Q \in \mathrm{U\Psi DO}^{-k}(E)$ be a parametrix for P, i.e., $PQ = \mathrm{id} - S_1$ and $QP = \mathrm{id} - S_2$, where $S_1, S_2 \in \mathrm{U\Psi DO}^{-\infty}(E)$. Integration by parts n times yields:

$$(iQ)^n \int_{\mathbb{R}} q^{(n)}(t)e^{itP}dt = (iQ)^n(-iP)^n \int_{\mathbb{R}} q(t)e^{itP}dt = (\mathrm{id} - S_2)^n \int_{\mathbb{R}} q(t)e^{itP}dt. \tag{2.7}$$

Since $q(t)|t|$ and $q^{(n)}(t)|t|$ are integrable and due to the Estimate (2.6), we conclude with Lemma 2.43 that both integrals $\int_{\mathbb{R}} q(t)e^{itP}dt$ and $\int_{\mathbb{R}} q^{(n)}(t)e^{itP}dt$ define quasilocal operators of order k on $H^{lk}(E)$. Note that for $\int_{\mathbb{R}} q(t)e^{itP}dt$ this is just a first result which we will need now in order to show that the order of this operator is in fact lower.

Now $(\mathrm{id} - S_2)^n = \mathrm{id} + \sum_{j=1}^n \binom{n}{j}(-S_2)^j$ and the sum is a quasilocal smoothing operator because S_2 is one. Since the composition of quasilocal operators is again a quasilocal operator (see [Roe88a, Proposition 5.2]), we conclude that the second summand R of

$$(\mathrm{id} - S_2)^n \int_{\mathbb{R}} q(t)e^{itP}dt = \int_{\mathbb{R}} q(t)e^{itP}dt + \underbrace{\sum_{j=1}^n \binom{n}{j}(-S_2)^j \int_{\mathbb{R}} q(t)e^{itP}dt}_{=:R} \tag{2.8}$$

is also a quasilocal smoothing operator. Now Equations (2.7) and (2.8) together yield

$$\int_{\mathbb{R}} q(t)e^{itP}dt = (iQ)^n \int_{\mathbb{R}} q^{(n)}(t)e^{itP}dt - R,$$

from which the claim follows. □

Recall that if f is a Schwartz function, then the operator $f(P)$ is given by

$$f(P) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t)e^{itP}dt, \tag{2.9}$$

where $\hat{f}$ is the Fourier transform of f. Since $\hat{f}$ is also a Schwartz function, it satisfies the assumption in Corollary 2.44 for all $n \in \mathbb{N}_0$, i.e., $f(P)$ is a quasilocal smoothing operator. Applying this argument to the adjoint operator $f(P)^* = \overline{f}(P)$, we get with Lemma 2.22 our next corollary:

Corollary 2.45. *If f is a Schwartz function, then $f(P) \in \mathrm{U\Psi DO}^{-\infty}(E)$.*

Recall from [Špa09, Lemma 4.2] that the uniformly pseudolocal operators form a C^*-algebra and that the uniformly locally compact operators form a closed, two-sided *-ideal in there. Since Schwartz functions are dense in $C_0(\mathbb{R})$ and quasilocal smoothing operators are uniformly locally compact (Corollary 2.30), we get with the above corollary that $g(P)$ is uniformly locally compact if $g \in C_0(\mathbb{R})$.

Corollary 2.46. *Let $g \in C_0(\mathbb{R})$. Then $g(P)$ is uniformly locally compact.*

Now we turn our attention to functions which are more general than Schwartz functions. To be concrete, we consider functions of the following type:

Definition 2.47 (Symbols on $\mathbb{R}$)**.** For arbitrary $m \in \mathbb{Z}$ we define

$$\mathcal{S}^m(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) \mid |f^{(n)}(x)| < C_n(1+|x|)^{m-n} \text{ for all } n \in \mathbb{N}_0\}.$$

Note that we have $\mathcal{S}(\mathbb{R}) = \bigcap_m \mathcal{S}^m(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space.

Let us state now the generalization of [Roe88a, Theorem 5.5] from operators of Dirac type to pseudodifferential operators:

Proposition 2.48 (cf. [Roe88a, Theorem 5.5])**.** *Let $f \in \mathcal{S}^m(\mathbb{R})$. Then for all $l \in \mathbb{Z}$ the operator $f(P)$ is a quasilocal operator of order mk on the Sobolev spaces $H^{lk}(E)$, i.e., $f(P)\colon H^{lk}(E) \to H^{lk-mk}(E)$.*

The proof of it is analogous but more technical since the operators e^{itP} are only quasilocal (Lemma 2.43) contrary to the operators e^{itD} which have finite propagation (see, e.g., [Roe88a, Theorem 1.3]). Moreover, we need Corollary 2.44 and the techniques developed in its proof for the adaption of [Roe88a, Theorem 5.5] to our case.

At last, let us turn our attention to a result regarding differences $\psi(P) - \psi(P')$ of operators defined via functional calculus. We will need the following proposition in the proof of the proposition where we show that elliptic pseudodifferential operators with the same symbol define the same uniform K-homology class.

Proposition 2.49 ([HR00, Proposition 10.3.7][30])**.** *Let ψ be a bounded Borel function whose distributional Fourier transform $\hat{\psi}$ is such that the product $s\hat{\psi}(s)$ is in $L^1(\mathbb{R})$.*

If P and P' are symmetric and elliptic pseudodifferential operators of positive order $k \geq 1$ such that their difference $P - P'$ has order qk, then we have for all $l \in \mathbb{Z}$

$$\|\psi(P) - \psi(P')\|_{lk,lk-qk} \leq C_\psi \cdot \|P - P'\|_{lk,lk-qk},$$

where the constant $C_\psi = \frac{1}{2\pi}\int |s\hat{\psi}(s)|ds$ does not depend on the operators.

Proof. We first assume that $\hat{\psi}$ is compactly supported and that $s\hat{\psi}(s)$ is a smooth function. Then we use the result [HR00, Proposition 10.3.5][31], which is a generalization of Equation 2.9 to more general functions than Schwartz functions, and get

$$\Big\langle \big(\psi(P) - \psi(P')\big)u, v \Big\rangle_{H^{lk-qk}} = \frac{1}{2\pi}\int \Big\langle \big(e^{isP} - e^{isP'}\big)u, v\Big\rangle_{H^{lk-qk}} \cdot \hat{\psi}(s)ds,$$

[30] The cited proposition requires additionally a common invariant domain for P and P'. In our case here this domain is given by, e.g., $H^\infty(E)$.

[31] Though stated there only for differential operators, its proof also works word-for-word for pseudodifferential ones.

for all $u, v \in C_c^\infty(E)$. From the Fundamental Theorem of Calculus we get

$$\Big\langle \big(e^{isP} - e^{isP'}\big)u, v\Big\rangle_{H^{lk-qk}} = i \cdot \int_0^s \Big\langle \big(e^{itP}(P - P')e^{i(s-t)P'}\big)u, v\Big\rangle_{H^{lk-qk}} dt$$

and therefore

$$\Big|\Big\langle \big(e^{isP} - e^{isP'}\big)u, v\Big\rangle_{H^{lk-qk}}\Big| \leq s \cdot \|P - P'\|_{lk,lk-qk} \cdot \|u\|_{lk} \cdot \|v\|_{lk-qk}.$$

Putting it all together, we get

$$\Big|\Big\langle \big(\psi(P) - \psi(P')\big)u, v\Big\rangle_{H^{lk-qk}}\Big| \leq C_\psi \cdot \|P - P'\|_{lk,lk-qk} \cdot \|u\|_{lk} \cdot \|v\|_{lk-qk}.$$

Now the general claim follows from an approximation argument analogous as at the end of the proof of [HR00, Proposition 10.3.5]. □

3 Uniform K-homology

Since we are considering uniform pseudodifferential operators, we need a K-homology theory that incorporates into its definition this uniformity. Such a theory was introduced by Špakula and the goal of this section if to revisit it and to prove vertain properties (existence of the Kasparov product and deducing from it homotopy invariance of uniform K-homology) that we will crucially need later and which were not proved by Špakula. Furthermore, we will use in Section 3.5 homotopy invariace to deduce useful facts about the rough Baum–Connes assembly map.

3.1 Definition and basic properties of uniform K-homology

Let us first recall briefly the notion of multigraded Hilbert spaces.

A *graded Hilbert space* is a Hilbert space H with a decomposition $H = H^+ \oplus H^-$ into closed, orthogonal subspaces. This is equivalent to the existence of a *grading operator* ϵ such that its ± 1-eigenspaces are exactly $H^\pm$ and such that ϵ is a selfadjoint unitary.

If H is a graded space, then its *opposite* is the graded space H^{op} whose underlying vector space is H, but with the reversed grading, i.e., $(H^{\mathrm{op}})^+ = H^-$ and $(H^{\mathrm{op}})^- = H^+$. This is equivalent to $\epsilon_{H^{\mathrm{op}}} = -\epsilon_H$.

An operator on a graded space H is called *even* if it maps $H^\pm$ again to $H^\pm$, and it is called *odd* if it maps $H^\pm$ to $H^\mp$. Equivalently, an operator is even if it commutes with the grading operator ϵ of H, and it is odd if it anti-commutes with it.

Definition 3.1 (Multigraded Hilbert spaces)**.** Let $p \in \mathbb{N}_0$. A *p-multigraded Hilbert space* is a graded Hilbert space which is equipped with p odd unitary operators $\epsilon_1, \ldots, \epsilon_p$ such that $\epsilon_i\epsilon_j + \epsilon_j\epsilon_i = 0$ for $i \neq j$, and $\epsilon_j^2 = -1$ for all j.

Note that a 0-multigraded Hilbert space is just a graded Hilbert space. We make the convention that a (-1)-multigraded Hilbert space is an ungraded one.

Definition 3.2 (Multigraded operators)**.** Let H be a p-multigraded Hilbert space. Then an operator on H will be called *multigraded*, if it commutes with the multigrading operators $\epsilon_1, \ldots, \epsilon_p$ of H.

Let us now recall the usual definition of multigraded Fredholm modules, where X is a locally compact, separable metric space:

Definition 3.3 (Multigraded Fredholm modules)**.** Let $p \in \mathbb{Z}_{\geq -1}$. A *$p$-multigraded Fredholm module (H, ρ, T) over X* is given by the following data:

- a separable p-multigraded Hilbert space H,
- a representation $\rho\colon C_0(X) \to \mathfrak{B}(H)$ by even, multigraded operators and
- an odd multigraded operator $T \in \mathfrak{B}(H)$ such that
  - the operators $T^2 - 1$ and $T - T^*$ are locally compact and
  - the operator T itself is pseudolocal.

  Here an operator S is called *locally compact*, if for all $f \in C_0(X)$ the operators $\rho(f)S$ and $S\rho(f)$ are compact, and S is called *pseudolocal*, if for all $f \in C_0(X)$ the operator $[S, \rho(f)]$ is compact.

Let us define

$$L\text{-}\operatorname{Lip}_R(X) := \{f \in C_c(X) \mid f \text{ is } L\text{-Lipschitz}, \operatorname{diam}(\operatorname{supp} f) \leq R \text{ and } \|f\|_\infty \leq 1\}.$$

Definition 3.4 ([Špa09, Definition 2.3])**.** Let $T \in \mathfrak{B}(H)$ be an operator on a Hilbert space H and $\rho\colon C_0(X) \to \mathfrak{B}(H)$ a representation.

We say that T is *uniformly locally compact*, if for every $R, L > 0$ the collection

$$\{\rho(f)T, T\rho(f) \mid f \in L\text{-}\operatorname{Lip}_R(X)\}$$

is uniformly approximable (Definition 2.25).

We say that T is *uniformly pseudolocal*, if for every $R, L > 0$ the collection

$$\{[T, \rho(f)] \mid f \in L\text{-}\operatorname{Lip}_R(X)\}$$

is uniformly approximable.

Note that by an approximation argument we get that the above defined collections are still uniformly approximable if we enlargen the definition of $L\text{-}\operatorname{Lip}_R(X)$ from $f \in C_c(X)$ to $f \in C_0(X)$.

The following lemma states that on proper spaces we may drop the L-dependence for uniformly locally compact operators.

Lemma 3.5 ([Špa09, Remark 2.5])**.** *Let X be a proper space. If T is uniformly locally compact, then for every $R > 0$ the collection*

$$\{\rho(f)T, T\rho(f) \mid f \in C_c(X), \operatorname{diam}(\operatorname{supp} f) \leq R \text{ and } \|f\|_\infty \leq 1\}$$

is also uniformly approximable (i.e., we can drop the L-dependence).

Note that an analogous lemma for uniformly pseudolocal operators does not hold. We may see this via the following example: if we have an operator D of Dirac type on a manifold M and if g is a smooth function on M, then we have the equation $([D,g]u)(x) = \sigma_D(x, dg)u(x)$, where u is a section into the Dirac bundle S on which D acts, $\sigma_D(x,\xi)$ is the symbol of D regarded as an endomorphism of S_x and $\xi \in T_x^*M$. So we see that the norm of $[D,g]$ does depend on the first derivative of the function g.

Definition 3.6 (Uniform Fredholm modules, cf. [Špa09, Definition 2.6])**.** A Fredholm module (H, ρ, T) is called *uniform*, if T is uniformly pseudolocal and the operators $T^2 - 1$ and $T - T^*$ are uniformly locally compact.

For a totally bounded metric space uniform Fredholm modules are the same as usual Fredholm modules. Since Špakula does not give a proof of it, we will do it now:

Proposition 3.7. *Let X be a totally bounded metric space. Then every Fredholm module over X is uniform.*

Proof. Let (H, ρ, T) be a Fredholm module.

First we will show that T is uniformly pseudolocal. We will use the fact that the set $L\text{-}\operatorname{Lip}_R(X) \subset C(X)$ is relatively compact (i.e., its closure is compact) by the Theorem of Arzelà–Ascoli.[32] Assume that T is not uniformly pseudolocal. Then there would be $R, L > 0$ and $\varepsilon > 0$, so that for all $N > 0$ we would have an $f_N \in L\text{-}\operatorname{Lip}_R(X)$ such that for all rank-N operators k we have $\|[T, \rho(f_N)] - k\| \geq \varepsilon$. Since $L\text{-}\operatorname{Lip}_R(X)$ is relatively compact, the sequence f_N has an accumulation point $f_\infty \in L\text{-}\operatorname{Lip}_R(X)$. Then we have $\|[T, \rho(f_\infty)] - k\| \geq \varepsilon/2$ for all finite rank operators k, which is a contradiction.

The proofs that $T^2 - 1$ and $T - T^*$ are uniformly locally compact are analogous. $\square$

A collection (H, ρ, T_t) of uniform Fredholm modules is called an *operator homotopy* if $t \mapsto T_t \in \mathfrak{B}(H)$ is norm continuous. As in the non-uniform case, we have an analogous lemma about compact perturbations:

Lemma 3.8 (Compact perturbations, cf. [Špa09, Lemma 2.16])**.** *Let (H, ρ, T) be a uniform Fredholm module and $K \in \mathfrak{B}(H)$ a uniformly locally compact operator.*

Then (H, ρ, T) and $(H, \rho, T + K)$ are operator homotopic.

Definition 3.9 (Uniform K-homology, cf. [Špa09, Definition 2.13])**.** We define the *uniform K-homology group $K_p^u(X)$* of a locally compact and separable metric space X to be the abelian group generated by unitary equivalence classes of p-multigraded uniform Fredholm modules with the relations:

- if x and y are operator homotopic, then $[x] = [y]$, and
- $[x] + [y] = [x \oplus y]$,

where x and y are p-multigraded uniform Fredholm modules.

[32] Since Lipschitz functions are uniformly continuous they have a unique extension to the completion $\overline{X}$ of X. Since $\overline{X}$ is compact, Arzelà–Ascoli applies.

All the basic properties of usual K-homology do also hold for uniform K-homology (e.g., that degenerate uniform Fredholm modules represent the zero class, that we have formal 2-periodicity $K_p^u(X) \cong K_{p+2}^u(X)$ for all $p \geq -1$, etc.).

For discussing functoriality of uniform K-homology we need the following definition:

Definition 3.10 (Uniformly cobounded maps, [Špa09, Definition 2.15])**.** Let us call a map $g\colon X \to Y$ with the property

$$\sup_{y \in Y} \operatorname{diam}(g^{-1}(B_r(y))) < \infty \text{ for all } r > 0$$

uniformly cobounded[33].

Note that if X is proper, then every uniformly cobounded map is proper (i.e., preimages of compact subsets are compact).

The following lemma about functoriality of uniform K-homology was proved by Špakula (see the paragraph directly after [Špa09, Definition 2.15]).

Lemma 3.11. *Uniform K-homology is functorial with respect to uniformly cobounded, proper Lipschitz maps, i.e., if $g\colon X \to Y$ is uniformly cobounded, proper and Lipschitz, then it induces maps $g_*\colon K_*^u(X) \to K_*^u(Y)$ on uniform K-homology via*

$$g_*[(H, \rho, T)] := [(H, \rho \circ g^*, T)],$$

where $g^*\colon C_0(Y) \to C_0(X)$, $f \mapsto f \circ g$ is the by g induced map on functions.

Recall that K-homology may be normalized in various ways, i.e., we may assume that the Fredholm modules have a certain form or a certain property and that this holds also for all homotopies.

Combining Lemmas 4.5 and 4.6 and Proposition 4.9 from [Špa09], we get the following:

Lemma 3.12. *We can normalize uniform K-homology $K_*^u(X)$ to involutive modules.*

The proof of the following Lemma 3.13 in the non-uniform case may be found in, e.g., [HR00, Lemma 8.3.8]. The proof in the uniform case is analogous and the arguments similar to the ones in the proofs of [Špa09, Lemmas 4.5 & 4.6].

Lemma 3.13. *Uniform K-homology $K_*^u(X)$ may be normalized to* non-degenerate *Fredholm modules, i.e., such that all occuring representations ρ are non-degenerate*[34].

Note that in general we can not normalize uniform K-homology to be simultaneously involutive and non-degenerate, just as usual K-homology.

Later we will also have to normalize Fredholm modules to finite propagation. But this is not always possible if the underlying metric space X is badly behaved. Therefore we get now to the definition of bounded geometry for metric spaces.

[33] Block and Weinberger call this property *effectively proper* in [BW92]. The author called it *uniformly proper* in his thesis [Eng14].

[34] This means that $\rho(C_0(X))H$ is dense in H.

Definition 3.14 (Coarsely bounded geometry)**.** Let X be a metric space. We call a subset $\Gamma \subset X$ a *quasi-lattice* if

- there is a $c > 0$ such that $B_c(\Gamma) = X$ (i.e., Γ is *coarsely dense*) and
- for all $r > 0$ there is a $K_r > 0$ such that $\#(\Gamma \cap B_r(y)) \leq K_r$ for all $y \in X$.

A metric space is said to have *coarsely bounded geometry*[35] if it admits a quasi-lattice.

Note that if we have a quasi-lattice $\Gamma \subset X$, then there also exists a uniformly discrete quasi-lattice $\Gamma' \subset X$. The proof of this is an easy application of the Lemma of Zorn: given an arbitrary $\delta > 0$ we look at the family $\mathcal{A}$ of all subsets $A \subset \Gamma$ with $d(x, y) > \delta$ for all $x, y \in A$. These subsets are partially ordered under inclusion of sets and every totally ordered chain $A_1 \subset A_2 \subset \ldots \subset \Gamma$ has an upper bound given by the union $\bigcup_i A_i \in \mathcal{A}$. So the Lemma of Zorn provides us with a maximal element $\Gamma' \in \mathcal{A}$. That Γ' is a quasi-lattice follows from its maximality.

Examples 3.15. Every Riemannian manifold M of bounded geometry[36] is a metric space of coarsely bounded geometry: any maximal set $\Gamma \subset M$ of points which are at least a fixed distance apart (i.e., there is an $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for all $x \neq y \in \Gamma$) will do the job. We can get such a maximal set by invoking Zorn's lemma. Note that a manifold of bounded geometry will also have locally bounded geometry, so no confusion can arise by not distinguishing between "coarsely" and "locally" bounded geometry in the terminology for manifolds.

If (X, d) is an arbitrary metric space that is bounded, i.e., $d(x, x') < D$ for all $x, x' \in X$ and some D, then *any* finite subset of X will constitute a quasi-lattice.

Let K be a simplicial complex of bounded geometry[37]. Equipping K with the metric derived from barycentric coordinates, the set of all vertices becomes a quasi-lattice in K.

If X has coarsely bounded geometry it will be crucial for us that we can normalize uniform K-homology to uniform finite propagation, i.e., such there is an $R > 0$ depending only on X such that every uniform Fredholm module has propagation at most R[38]. This was proved by Špakula in [Špa09, Proposition 7.4]. Note that it is in general not possible to make this common propagation R arbitrarily small. Furthermore, we can combine the normalization to finite propagation with the other normalizations.

Proposition 3.16 ([Špa09, Section 7])**.** *If X has coarsely bounded geometry, then there is an $R > 0$ depending only on X such that uniform K-homology may be normalized to uniform Fredholm modules that have propagation at most R.*

Furthermore, we can additionally normalize them to either involutive modules or to non-degenerate ones.

[35]Note that most authors call this property just "bounded geometry". But since later we will also have the notion of locally bounded geometry, we use for this one the term "coarsely" to distinguish them.
[36]That is to say, the injectivity radius of M is uniformly positive and the curvature tensor and all its derivatives are bounded in sup-norm.
[37]That is, the number of simplices in the link of each vertex is uniformly bounded.
[38]This means $\rho(f)T\rho(g) = 0$ if $d(\operatorname{supp} f, \operatorname{supp} g) > R$.

Having discussed the normalization to finite propagation modules, we can now compute an easy but important example:

Lemma 3.17. *Let Y be a uniformly discrete, proper metric space of coarsely bounded geometry. Then $K_0^u(Y)$ is isomorphic to the group $\ell^\infty_{\mathbb{Z}}(Y)$ of all bounded, integer-valued sequences indexed by Y, and $K_1^u(Y) = 0$.*

Proof. We use Proposition 3.16 to normalize uniform K-homology to operators of finite propagation, i.e., there is an $R > 0$ such that every uniform Fredholm module over Y may be represented by a module (H, ρ, T) where T has propagation no more than R and all homotopies may be also represented by homotopies where the operators have propagation at most R.

Going into the proof of Proposition 3.16, we see that in our case of a uniformly discrete metric space Y we may choose R less than the least distance between two points of Y, i.e., $0 < R < \inf_{x \neq y \in Y} d(x, y)$. So given a module (H, ρ, T) where T has propagation at most R, the operator T decomposes as a direct sum $T = \bigoplus_{y \in Y} T_y$ with $T_y \colon H_y \to H_y$. The Hilbert space H_y is defined as $H_y := \rho(\chi_y)H$, where χ_y is the characteristic function of the single point $y \in Y$. Note that χ_y is a continuous function since the space Y is discrete. Hence $(H, \rho, T) = \bigoplus (H_y, \rho_y, T_y)$ with $\rho_y \colon C_0(Y) \to \mathfrak{B}(H_y)$, $f \mapsto \rho(\chi_y)\rho(f)\rho(\chi_y)$. Now each (H_y, ρ_y, T_y) is a Fredholm module over the point y and so we get a map

$$K_*^u(Y) \to \prod_{y \in Y} K_*^u(y).$$

Note that we need that the homotopies also all have propagation at most R so that the above defined decomposition of a uniform Fredholm module descends to the level of uniform K-homology.

Since a point y is for itself a compact space, we have $K_*^u(y) = K_*(y)$, and the latter group is isomorphic to $\mathbb{Z}$ for $* = 0$ and it is 0 for $* = 1$. Since the above map $K_*^u(Y) \to \prod_{y \in Y} K_*^u(y)$ is injective, we immediately conclude $K_1^u(Y) = 0$.

So it remains to show that the image of this map in the case $* = 0$ consists of the *bounded* integer-valued sequences indexed by Y. But this follows from the uniformity condition in the definition of uniform K-homology: the isomorphism $K_0(y) \cong \mathbb{Z}$ is given by assigning a module (H_y, ρ_y, T_y) the Fredholm index of T (note that T_y is a Fredholm operator since (H_y, ρ_y, T_y) is a module over a single point). Now since $(H, \rho, T) = \bigoplus (H_y, \rho_y, T_y)$ is a *uniform* Fredholm module, we may conclude that the Fredholm indices of the single operators T_y are bounded with respect to y. □

3.2 Differences to Špakula's version

We will discuss now the differences between our version of uniform K-homology and Špakula's version from [Špa08] and [Špa09].

Firstly, our definition of uniform K-homology is based on multigraded Fredholm modules and we therefore have groups $K_p^*(X)$ for all $p \geq -1$, but Špakula only defined K_0^u and K_1^u. This is not a real restriction since uniform K-homology has, analogously as

usual K-homology, a formal 2-periodicity, i.e., there are only two essentially different groups: K_0^u and K_1^u. We mention this since if the reader wants to look up the original reference [Špa08] and [Špa09], he has to keep in mind that we work with multigraded modules, but Špakula does not.

Secondly, Špakula gives the definition of uniform K-homology only for proper[39] metric spaces. The reason for this is that certain results of him (Sections 8-9 in [Špa09]) only work for proper spaces. This results are all connected to the rough assembly map $\mu_u \colon K_*^u(X) \to K_*(C_u^*(Y))$, where $Y \subset X$ is a uniformly discrete quasi-lattice, and this is nor surprising: the (uniform) Roe algebra only has on proper spaces nice properties (like its K-theory being a coarse invariant) and therefore we expect that results of uniform K-homology that connect to the uniform Roe algebra also should need the properness assumptions on the space X. But we can see by looking into the proofs of Špakula in all the other sections of [Špa09] that all results except the ones in Sections 8-9 also hold for locally compact, separable metric spaces (without assumptions on completeness or properness). Note that this is a very crucial fact for us that uniform K-homology does also make sense for non-proper spaces since in the proof of Poincaré duality we will have to consider the uniform K-homology of open balls in $\mathbb{R}^n$.

Thirdly, Špakula uses the notion "L-continuous" instead of "L-Lipschitz" for the definition of $L\text{-}\operatorname{Lip}_R(X)$ (which he also denotes by $C_{R,L}(X)$, i.e., we have also changed the notation), so that he gets slightly differently defined uniform Fredholm modules. But the author was not able to deduce Proposition 3.7 with Špakula's definition, which is why we have changed it to "L-Lipschitz" (since the statement of Proposition 3.7 is a very desirable one and, in fact, we will need it crucially in the proof of Poincaré duality). Špakula noted that for a geodesic metric space both notions (L-continuous and L-Lipschitz) coincide, i.e., for probably all spaces which one wants to consider ours and Špakula's definition coincide. But note that all the results of Špakula do also hold with our definition of uniform Fredholm modules, i.e., changing the definition to ours does not affect the validity of his results.

And last, let us get to the most crucial difference: to define uniform K-homology Špakula does not use operator homotopy as a relation but a weaker form of homotopy ([Špa09, Definition 2.11]). The reasons why we changed this are the following: firstly, the definition of usual K-homology uses operator homotopy and it seems desirable to have uniform K-homology to be similarly defined, i.e., just imposing an additional condition on the Fredholm modules. Secondly, Špakula's proof of [Špa09, Proposition 4.9] is not correct under his notion of homotopy, but it becomes correct if we use operator homotopy as a relation. So by changing the definition we ensure that [Špa09, Proposition 4.9] does hold. And thirdly, we prove in Section 3.4 that we would get the same uniform K-homology groups if we impose weak homotopy (Definition 3.27) as a relation instead of operator homotopy. Though our notion of weak homotopies is different from Špakula's notion of homotopies, all the homotopies that he constructs in his paper [Špa09] are weak homotopies, i.e., all the results of him that rely on his notion of homotopy are also true with our definition.

[39] That means that all closed balls are compact.

To put it into a nutshell, we changed the definition of uniform K-homology in order to make the definition similar to one of usual K-homology and to correct Špakula's proof of [Špa09, Proposition 4.9]. It also seems to be easier to work with our version. Furthermore, all of his results do also hold in our definition. And last, we remark that his results, besides the ones in Sections 8-9 in [Špa09], also hold for non-proper, non-complete spaces.

3.3 External product

Now we get to one of the most important technical parts in this article: the construction of the external product for uniform K-homology. Its main application will be to deduce homotopy invariance of uniform K-homology.

Note that we can construct the product only if the involved metric spaces have jointly bounded geometry (which we will define in a moment), and since this property is crucially used, the author does not see any way to overcome this requirement. But fortunately, both major classes of spaces on which we want to apply our theory, namely manifolds and simplicial complexes of bounded geometry, do have jointly bounded geometry.

Definition 3.18 (Locally bounded geometry, [Špa10, Definition 3.1])**.** A metric space X has *locally bounded geometry*, if it admits a countable Borel decomposition $X = \cup X_i$ such that

- each X_i has non-empty interior,
- each X_i is totally bounded, and
- for all $\varepsilon > 0$ there is an $N > 0$ such that for every X_i there exists an ε-net in X_i of cardinality at most N.

Note that Špakula demands in his definition of "locally bounded geometry" that the closure of each X_i is compact instead of the total boundedness of them. The reason for this is that he considers only proper spaces, whereas we need a more general notion to encompass also non-complete spaces.

Definition 3.19 (Jointly bounded geometry)**.** A metric space X has *jointly coarsely and locally bounded geometry*, if

- it admits a countable Borel decomposition $X = \cup X_i$ satisfying all the properties of the above Definition 3.18 of locally bounded geometry,
- it admits a quasi-lattice $\Gamma \subset X$ (i.e., X has coarsely bounded geometry), and
- for all $r > 0$ we have $\sup_{y \in \Gamma} \#\{i \mid B_r(y) \cap X_i \neq \emptyset\} < \infty$.

The last property ensures that there is an upper bound on the number of subsets X_i that intersect any ball of radius $r > 0$ in X.

Examples 3.20. Recall from Examples 3.15 that manifolds of bounded geometry and simplicial complexes of bounded geometry (i.e., the number of simplices in the link of each vertex is uniformly bounded) equipped with the metric derived from barycentric coordinates have coarsely bounded geometry. Now a moment of reflection reveals that they even have jointly bounded geometry.

In the next Figure 1 we give an example of a space X having coarsely and locally bounded geometry, but where the quasi-lattice Γ and the Borel decomposition $X = \cup X_i$ are not compatible with each other:

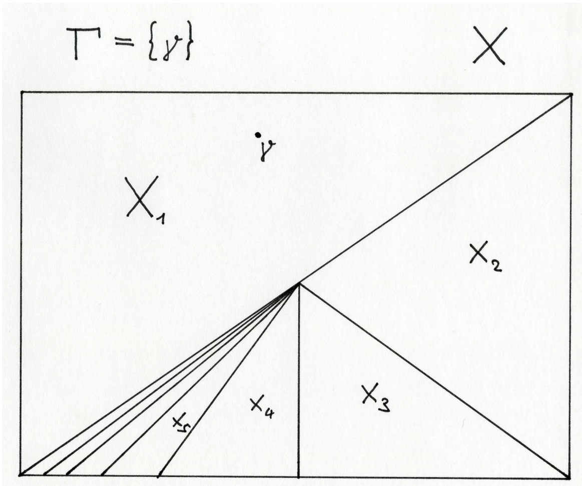

Figure 1: Coarsely and locally bounded geometry, but they are not compatible.

In our construction of the product for uniform K-homology we follow the presentation in [HR00, Section 9.2], where the product is constructed for usual K-homology.

Let X_1 and X_2 be locally compact and separable metric spaces and both having jointly bounded geometry, (H_1, ρ_1, T_1) a p_1-multigraded uniform Fredholm module over the space X_1 and (H_2, ρ_2, T_2) a p_2-multigraded module over X_2, and both modules will be assumed to have finite propagation (see Proposition 3.16).

Definition 3.21 (cf. [HR00, Definition 9.2.2])**.** We define ρ to be the tensor product representation of $C_0(X_1 \times X_2) \cong C_0(X_1) \otimes C_0(X_2)$ on $H := H_1 \hat{\otimes} H_2$, i.e.,

$$\rho(f_1 \otimes f_2) = \rho_1(f_1) \hat{\otimes} \rho_2(f_2) \in \mathfrak{B}(H_1) \hat{\otimes} \mathfrak{B}(H_2)$$

and equip $H_1 \hat{\otimes} H_2$ with the induced $(p_1 + p_2)$-multigrading[40].

[40] The graded tensor product $H_1 \hat{\otimes} H_2$ is $(p_1 + p_2)$-multigraded if we let the multigrading operators ϵ_j of H_1 act on the tensor product as

$$\epsilon_j(v_1 \otimes v_2) := (-1)^{\deg(v_2)} \epsilon_j(v_1) \otimes v_2$$

for $1 \le j \le p_1$, and for $1 \le j \le p_2$ we let the multigrading operators ϵ_{p_1+j} of H_2 act as

$$\epsilon_{p_1+j}(v_1 \otimes v_2) := v_1 \otimes \epsilon_{p_1+j}(v_2).$$

We say that a (p_1+p_2)-multigraded uniform Fredholm module (H,ρ,T) is *aligned* with the modules (H_1,ρ_1,T_1) and (H_2,ρ_2,T_2), if

- T has finite propagation,
- for all $f \in C_0(X_1\times X_2)$ the operators
$$\rho(f)\big(T(T_1\hat\otimes 1)+(T_1\hat\otimes 1)T\big)\rho(\bar f) \text{ and } \rho(f)\big(T(1\hat\otimes T_2)+(1\hat\otimes T_2)T\big)\rho(\bar f)$$
are positive modulo compact operators,[41] and
- for all $f\in C_0(X_1\times X_2)$ the operator $\rho(f)T$ derives $\mathfrak{K}(H_1)\hat\otimes\mathfrak{B}(H_2)$, i.e.,
$$[\rho(f)T,\mathfrak{K}(H_1)\hat\otimes\mathfrak{B}(H_2)]\subset\mathfrak{K}(H_1)\hat\otimes\mathfrak{B}(H_2).$$

Since both H and ρ are uniquely determined from H_1, ρ_1, H_2 and ρ_2, we will often just say that T *is aligned with* T_1 *and* T_2.

Our major technical lemma is the following one. It is a uniform version of Kasparov's Technical Lemma, which is suitable for our needs.

Lemma 3.22. *Let X_1 and X_2 be locally compact and separable metric spaces that have jointly coarsely and locally bounded geometry.*

Then there exist commuting, even, multigraded, positive operators N_1, N_2 of finite propagation on $H:=H_1\hat\otimes H_2$ with $N_1^2+N_2^2=1$ and the following properties:

1. *$N_1\cdot\big\{(T_1^2-1)\rho_1(f)\hat\otimes 1 \mid f\in L\text{-}\mathrm{Lip}_{R'}(X_1)\big\}\subset\mathfrak{K}(H_1\hat\otimes H_2)$ is uniformly approximable for all $R',L>0$ and analogously for $(T_1^*-T_1)\rho_1(f)$ and for $[T_1,\rho_1(f)]$ instead of $(T_1^2-1)\rho_1(f)$,*
2. *$N_2\cdot\big\{1\hat\otimes(T_2^2-1)\rho_2(f)\mid f\in L\text{-}\mathrm{Lip}_{R'}(X_2)\big\}\subset\mathfrak{K}(H_1\hat\otimes H_2)$ is uniformly approximable for all $R',L>0$ and analogously for $(T_2^*-T_2)\rho_2(f)$ and for $[T_2,\rho_2(f)]$ instead of $(T_2^2-1)\rho_2(f)$,*
3. *$\{[N_i,T_1\hat\otimes 1]\rho(f),[N_i,1\hat\otimes T_2]\rho(f)\mid f\in L\text{-}\mathrm{Lip}_{R'}(X_1\times X_2)\}$ is uniformly approximable for all $R',L>0$ and both $i=1,2$,*
4. *$\big\{[N_i,\rho(f\otimes 1)],[N_i,\rho(1\otimes g)]\mid f\in L\text{-}\mathrm{Lip}_{R'}(X_1), g\in L\text{-}\mathrm{Lip}_{R'}(X_2)\big\}$ is uniformly approximable for all $R',L>0$ and both $i=1,2$, and*
5. *both N_1 and N_2 derive $\mathfrak{K}(H_1)\hat\otimes\mathfrak{B}(H_2)$.*

Proof. Due to the jointly bounded geometry there is a countable Borel decomposition $\{X_{1,i}\}$ of X_1 such that each $X_{1,i}$ has non-empty interior, the completions $\{\overline{X_{1,i}}\}$ form an admissible class[42] of compact metric spaces and for each $R>0$ we have
$$\sup_i \#\{j\mid B_R(X_{1,i})\cap X_{1,j}\neq\emptyset\}<\infty. \tag{3.1}$$

[41] That is to say, they are positive in the Calkin algebra $\mathfrak{B}(H)/\mathfrak{K}(H)$.

[42] This means that for every $\varepsilon>0$ there is an $N>0$ such that in every $\overline{X_{1,i}}$ exists an ε-net of cardinality at most N.

The completions of the 1-balls $B_1(X_{1,i})$ are also an admissible class of compact metric spaces and the collection of these open balls forms a uniformly locally finite open cover of X_1. We may find a partition of unity $\varphi_{1,i}$ subordinate to the cover $\{B_1(X_{1,i})\}$ such that every function $\varphi_{1,i}$ is L_0-Lipschitz for a fixed $L_0 > 0$ (but we will probably have to enlarge the value of L_0 a bit in a moment). The same holds also for a countable Borel decomposition $\{X_{2,i}\}$ of X_2 and we choose a partition of unity $\varphi_{2,i}$ subordinate to the cover $\{B_1(X_{2,i})\}$ such that every function $\varphi_{2,i}$ is also L_0-Lipschitz (by possibly enlargening L_0 so that we have the same Lipschitz constant for both partitions of unity).

Since $\{\overline{B_1(X_{1,i})}\}$ is an admissible class of compact metric spaces, we have for each $\varepsilon > 0$ and $L > 0$ a bound independent of i on the number of functions from

$$\varphi_{1,i} \cdot L\text{-}\operatorname{Lip}_c(X_1) := \{\varphi_{1,i} \cdot f \mid f \text{ is } L\text{-Lipschitz, compactly supported and } \|f\|_\infty \leq 1\}$$

to form an ε-net in $\varphi_{1,i} \cdot L\text{-}\operatorname{Lip}_c(X_1)$, and analogously for X_2 (this can be proved by a similar construction as the one from [Špa10, Lemma 2.4]). We denote this upper bound by $C_{\varepsilon,L}$.

Now for each $N \in \mathbb{N}$ and $i \in \mathbb{N}$ we choose $C_{1/N,N}$ functions $\{f_k^{i,N}\}_{k=1,\ldots,C_{1/N,N}}$ from $\varphi_{1,i} \cdot N\text{-}\operatorname{Lip}_c(X_{1,i})$ constituting an $1/N$-net.[43] Analogously we choose $C_{1/N,N}$ functions $\{g_k^{i,N}\}_{k=1,\ldots,C_{1/N,N}}$ from $\varphi_{2,i} \cdot N\text{-}\operatorname{Lip}_c(X_{2,i})$ that are $1/N$-nets.

We choose a sequence $\{u_n \hat{\otimes} 1\} \subset \mathfrak{B}(H_1) \hat{\otimes} \mathfrak{B}(H_2)$ of operators in the following way: u_n will be a projection operator onto a subspace U_n of H_1. To define this subspace, we first consider the operators

$$(T_1^2 - 1)\rho_1(f),\ (T_1 - T_1^*)\rho_1(f),\ \text{and } [T_1, \rho_1(f)] \tag{3.2}$$

for suitable functions $f \in C_0(X_1)$ that we will choose in a moment. These operators are elements of $\mathfrak{K}(H_1)$ since (H_1, ρ_1, T_1) is a Fredholm module. So up to an error of 2^{-n} they are of finite rank and the span V_n of the images of these finite rank operators will be the building block for the subspace U_n on which the operator u_n projects[44] (i.e., we will say in a moment how to enlarge V_n in order to get U_n). We choose the functions $f \in C_0(X_1)$ as all the functions from the set $\bigcup \{f_k^{i,N}\}_{k=1,\ldots,C_{1/N,N}}$, where the union ranges over all $i \in \mathbb{N}$ and $1 \leq N \leq n$. Note that since the Fredholm module (H_1, ρ_1, T_1) is uniform, the rank of the finite rank operators approximating (3.2) up to an error of 2^{-n} is bounded from above with a bound that depends only on N and n, but not on i nor k. Since we will have $V_n \subset U_n$, we can already give the first estimate that we will need later:

$$\|(u_n \hat{\otimes} 1)(x \hat{\otimes} 1) - (x \hat{\otimes} 1)\| < 2^{-n}, \tag{3.3}$$

[43] If we need less functions to get an $1/N$-net, we still choose $C_{1/N,N}$ of them. This makes things easier for us to write down.

[44] This finite rank operators are of course not unique. Recall that every compact operator on a Hilbert space H may be represented in the form $\sum_{n \geq 1} \lambda_n \langle f_n, \cdot \rangle g_n$, where the values λ_n are the singular values of the operator and $\{f_n\}$, $\{g_n\}$ are orthonormal (though not necessarily complete) families in H (but contrary to the λ_n they are not unique). Now we choose our finite rank operator to be the operator given by the same sum, but only with the λ_n satisfying $\lambda_n \geq 2^{-n}$.

where x is one of the operators from (3.2) for all $f_k^{i,N}$ with $1 \leq N \leq n$.[45] Moreover, denoting by $\chi_{1,i}$ the characteristic function of $B_1(X_{1,i})$, then $\rho_1(\chi_{1,i}) \cdot V_n$ is a subspace of H_1 of finite dimension that is bounded independently of i.[46] The reason for this is because T_1 has finite propagation and the number of functions $f_k^{i,N}$ for fixed N is bounded independently of i. For all n we also have $V_n \subset V_{n+1}$ and that the projection operator onto V_n has finite propagation which is bounded independently of n.

For each $n \in \mathbb{N}$ we partition $\chi_{1,i}$ for all $i \in \mathbb{N}$ into disjoint characteristic functions $\chi_{1,i} = \sum_{j=1}^{J_n} \chi_{1,i}^{j,n}$ such that we may write each function $f_k^{i,N}$ for all $i \in \mathbb{N}$, $1 \leq N \leq n$ and $k = 1, \ldots, C_{1/N,N}$ up to an error of 2^{-n-1} as a sum $f_k^{i,N} = \sum_{j=1}^{J_n} \alpha_k^{i,N}(j,n) \cdot \chi_{1,i}^{j,n}$ for suitable constants $\alpha_k^{i,N}(j,n)$. Note that since X_1 has jointly coarsely and locally bounded geometry, we can choose the upper bounds J_n such that they do not depend on i. Now we can finally set U_n as the linear span of V_n and $\rho_1(\chi_{1,i}^{j,n}) \cdot V_n$ for all $i \in \mathbb{N}$ and $1 \leq j \leq J_n$. Note that $\rho_1(\chi_{1,i}) \cdot U_n$ is a subspace of H_1 of finite dimension that is bounded independently of i, that we may choose the characteristic functions $\chi_{1,i}^{j,n}$ such that we have $U_n \subset U_{n+1}$ (by possibly enlargening each J_n), and that the projection operator u_n onto U_n has finite propagation which is bounded independently of n. Since we have $[u_n, \rho_1(\chi_{1,i}^{j,n})] = 0$ for all $i \in \mathbb{N}$, $1 \leq j \leq J_n$ and all $n \in \mathbb{N}$, we get our second crucial estimate:

$$\|[u_n \hat{\otimes} 1, \rho_1(f_k^{i,N}) \hat{\otimes} 1]\| < 2^{-n} \tag{3.4}$$

for all $i \in \mathbb{N}$, $k = 1, \ldots, C_{1/N,N}$, $1 \leq N \leq n$ and all $n \in \mathbb{N}$.

By an argument similar to the proof of the existence of quasicentral approximate units, we may conclude that for each $n \in \mathbb{N}$ there exists a finite convex combination ν_n of the elements $\{u_n, u_{n+1}, \ldots\}$ such that

$$\|[\nu_n \hat{\otimes} 1, T_1 \hat{\otimes} 1]\| < 2^{-n},\ \|[\nu_n \hat{\otimes} 1, \epsilon_1 \hat{\otimes} \epsilon_2]\| < 2^{-n} \text{ and } \|[\nu_n \hat{\otimes} 1, \epsilon^j]\| < 2^{-n} \tag{3.5}$$

for all $n \in \mathbb{N}$, where $\epsilon_1 \hat{\otimes} \epsilon_2$ is the grading operator of $H_1 \hat{\otimes} H_2$ and ϵ^j, $1 \leq j \leq p_1 + p_2$, are the multigrading operators of $H_1 \hat{\otimes} H_2$. Note that the Estimates (3.3) and (3.4) also hold for ν_n. Note furthermore that we can arrange that the maximal index occuring in the finite convex combination for ν_n is increasing in n.

Now we will construct a sequence $w_n \in \mathfrak{B}(H_1) \hat{\otimes} \mathfrak{B}(H_2)$ with suitable properties. We have that ν_n is a finite convex combination of the elements $\{u_n, u_{n+1}, \ldots\}$. So for $n \in \mathbb{N}$ we let m_n denote the maximal occuring index in that combination. Furthermore, we let the projections $p_n \in \mathfrak{B}(H_2)$ be analogously defined as u_n, where we consider now the operators

$$(T_2^2 - 1)\rho_2(g),\ (T_2 - T_2^*)\rho_2(g),\ \text{and } [T_2, \rho_2(g)] \tag{3.6}$$

for the analogous sets of functions $\bigcup \{g_k^{i,N}\}_{k=1,\ldots,C_{1/N,N}}$ depending on $n \in \mathbb{N}$. Then we

[45] Actually, to have this estimate we would need that x is self-adjoint. We can pass from x to $\frac{1}{2}(x + x^*)$ and $\frac{1}{2i}(x - x^*)$, do all the constructions with these self-adjoint operators and get the needed estimates for them, and then we get the same estimates for x but with an additional factor of 2.

[46] We have used here the fact that we may uniquely extend any representation of $C_0(Z)$ to one of the bounded Borel functions $B_b(Z)$ on a space Z.

define $w_{n-1} := u_{m_n} \hat{\otimes} p_n$[47] and get for all $n \in \mathbb{N}$ the following:

$$w_n(\nu_n \hat{\otimes} 1)(1 \hat{\otimes} p_n) = (\nu_n \hat{\otimes} 1)(1 \hat{\otimes} p_n) \tag{3.7}$$

and

$$\|[w_n, x \hat{\otimes} 1]\| < 2^{-n} \tag{3.8}$$
$$\|[w_n, 1 \hat{\otimes} y]\| < 2^{-n} \tag{3.9}$$
$$\|[w_n, \rho(f_k^{i,N} \otimes g_k^{i,N})]\| < 2^{-n} \tag{3.10}$$

for all $i \in \mathbb{N}$, $1 \le N \le n$ and $k = 1, \ldots, C_{1/N,N}$, where x is one of the operators from (3.2) for all $f_k^{i,N}$ and y is one of the operators from (3.6) for all $g_k^{i,N}$.

Let now $d_n := (w_n - w_{n-1})^{1/2}$. With a suitable index shift we can arrange that firstly, the Estimates (3.8)–(3.10) also hold for d_n instead of w_n,[48] and that secondly, using Equation (3.7),

$$\|d_n(\nu_n \hat{\otimes} 1)y\| < 2^{-n}, \tag{3.11}$$

where y is again one of the operators from (3.6) for all $g_k^{i,N}$ and $1 \le N \le n$.

Now as in the same way as we constructed ν_n out of the u_ns, we construct δ_n as a finite convex combination of the elements $\{d_n, d_{n+1}, \ldots\}$ such that

$$\|[\delta_n, T_1 \hat{\otimes} 1]\| < 2^{-n},\ \|[\delta_n, 1 \hat{\otimes} T_2]\| < 2^{-n},\ \|[\delta_n, \epsilon_1 \hat{\otimes} \epsilon_2]\| < 2^{-n} \text{ and } \|[\delta_n, \epsilon^j]\| < 2^{-n},$$

where $\epsilon_1 \hat{\otimes} \epsilon_2$ is the grading operator of $H_1 \hat{\otimes} H_2$ and ϵ^j for $1 \le j \le p_1 + p_2$ are the multigrading operators of $H_1 \hat{\otimes} H_2$. Clearly, all the Estimates (3.8)–(3.11) also hold for the operators δ_n.

Define $X := \sum \delta_n \nu_n \delta_n$. It is a positive operator of finite propagation and fulfills the Points 2–4 that N_2 should have. The arguments for this are analogous to the ones given at the end of the proof of [HR00, Kasparov's Technical Theorem 3.8.1], but we have to use all the uniform approximations that we additionally have (to use them, we have to cut functions $f \in L\text{-}\operatorname{Lip}_{R'}(X_1)$ down to the single "parts" $X_{1,i}$ of X_1 by using the partition of unity $\{\varphi_{1,i}\}$ that we have chosen at the beginning of this proof, and analogously for X_2). Furthermore, the operator $1 - X$ fulfills the desired Points 1, 3 and 4 that N_1 should fulfill. That both X and $1 - X$ derive $\mathfrak{K}(H_1) \hat{\otimes} \mathfrak{B}(H_2)$ is clear via construction. Since X commutes modulo compact operators with the grading and multigrading operators, we can average it over them so that it becomes an even and multigraded operator and X and $1 - X$ still have all the above mentioned properties.

Finally, we set $N_1 := (1 - X)^{1/2}$ and $N_2 := X^{1/2}$. $\square$

Now we will use this technical lemma to construct the external product and to show that it is well-defined on the level of uniform K-homology.

[47] The index is shifted by one so that we get the Estimates (3.8)–(3.10) with 2^{-n} and not with 2^{-n+1}; though this is not necessary for the argument.

[48] see [HR00, Exercise 3.9.6]

Proposition 3.23. *Let X_1 and X_2 be locally compact and separable metric spaces that have jointly coarsely and locally bounded geometry.*

Then there exists a (p_1+p_2)-multigraded uniform Fredholm module (H,ρ,T) which is aligned with the modules (H_1,ρ_1,T_1) and (H_2,ρ_2,T_2).

Furthermore, any two such aligned Fredholm modules are operator homotopic and this operator homotopy class is uniquely determined by the operator homotopy classes of (H_1,ρ_1,T_1) and (H_2,ρ_2,T_2).

Proof. We invoke the above Lemma 3.22 to get operators N_1 and N_2 and then set

$$T := N_1(T_1 \mathbin{\hat{\otimes}} 1) + N_2(1 \mathbin{\hat{\otimes}} T_2).$$

To deduce that (H,ρ,T) is a uniform Fredholm module, we have to use the following facts (additionally to the ones that N_1 and N_2 have): that T_1 and T_2 have finite propagation and are odd (we need that $(T_1 \mathbin{\hat{\otimes}} 1)(1 \mathbin{\hat{\otimes}} T_2) + (1 \mathbin{\hat{\otimes}} T_2)(T_1 \mathbin{\hat{\otimes}} 1) = 0$). To deduce that it is a multigraded module, we need that we constructed N_1 and N_2 as even and multigraded operators on H.

It is easily seen that for all $f \in C_0(X_1 \times X_2)$

$$\rho(f)\big(T(T_1 \mathbin{\hat{\otimes}} 1) + (T_1 \mathbin{\hat{\otimes}} 1)T\big)\rho(\bar{f}) \text{ and } \rho(f)\big(T(1 \mathbin{\hat{\otimes}} T_2) + (1 \mathbin{\hat{\otimes}} T_2)T\big)\rho(\bar{f})$$

are positive modulo compact operators and that $\rho(f)T$ derives $\mathfrak{K}(H_1) \mathbin{\hat{\otimes}} \mathfrak{B}(H_2)$, i.e., we conclude that T is aligned with T_1 and T_2.

Since all four operators T_1, T_2, N_1 and N_2 have finite propagation, T has also finite propagation.

Suppose that T' is another operator aligned with T_1 and T_2. We construct again operators N_1 and N_2 using the above Lemma 3.22, but we additionally enforce

$$\|[w_n, \rho(f_k^{i,N} \otimes g_k^{i,N})T']\| < 2^{-n}$$

analogously as we did it there to get Equation (3.10). So N_1 and N_2 will commute modulo compact operators with $\rho(f)T'$ for all functions $f \in C_0(X_1 \times X_2)$. Again, we set $T := N_1(T_1 \mathbin{\hat{\otimes}} 1) + N_2(1 \mathbin{\hat{\otimes}} T_2)$. Since N_1 and N_2 commute modulo compacts with $\rho(f)T'$ for all $f \in C_0(X_1 \times X_2)$ and since T' is aligned with T_1 and T_2, we conclude

$$\rho(f)(TT' + T'T)\rho(\bar{f}) \geq 0$$

modulo compact operators for all functions $f \in C_0(X_1 \times X_2)$. Using a uniform version of [HR00, Proposition 8.3.16] we conclude that T and T' are operator homotopic via multigraded, uniform Fredholm modules. We conclude that every aligned module is operator homotopic to one of the form that we constructed above, i.e., to one of the form $N_1(T_1 \mathbin{\hat{\otimes}} 1) + N_2(1 \mathbin{\hat{\otimes}} T_2)$. But all such operators are homotopic to one another: they are determined by the operator $Y = N_2^2$ used in the proof of the above lemma and the set of all operators with the same properties as Y is convex.

At last, suppose that one of the operators is varied by an operator homotopy, e.g., T_1 by $T_1(t)$. Then, in order to construct N_1 and N_2, we enforce in Equation (3.5) instead of $\|[\nu_n \mathbin{\hat{\otimes}} 1, T_1 \mathbin{\hat{\otimes}} 1]\| < 2^{-n}$ the following one:

$$\|[\nu_n \mathbin{\hat{\otimes}} 1, T_1(j/n) \mathbin{\hat{\otimes}} 1]\| < 2^{-n}$$

for $0 \leq j \leq n$. Now we may define

$$T(t) := N_1(T_1(t) \hat{\otimes} 1) + N_2(1 \hat{\otimes} T_2),$$

i.e., we got operators N_1 and N_2 which are independent of t but still have all the needed properties. This gives us the desired operator homotopy. □

Definition 3.24 (External product)**.** The *external product* of the multigraded uniform Fredholm modules (H_1, ρ_1, T_1) and (H_2, ρ_2, T_2) is a multigraded uniform Fredholm module (H, ρ, T) which is aligned with T_1 and T_2. We will use the notation $T := T_1 \times T_2$.

By the above Proposition 3.23 we know that if the locally compact and separable metric spaces X_1 and X_2 both have jointly coarsely and locally bounded geometry, then the external product always exists, that it is well-defined up to operator homotopy and that it descends to a well-defined product on the level of uniform K-homology:

$$K^u_{p_1}(X_1) \times K^u_{p_2}(X_2) \to K^u_{p_1+p_2}(X_1 \times X_2)$$

for $p_1, p_2 \geq 0$. Furthermore, this product is bilinear.[49]

For the remaining products (i.e., the product of an ungraded and a multigraded module, resp., the product of two ungraded modules) we can appeal to the formal 2-periodicity.

Associativity of the external product and the other important properties of it may be shown as in the non-uniform case. Let us summarize them in the following theorem:

Theorem 3.25 (External product for uniform K-homology)**.** *Let X_1 and X_2 be locally compact and separable metric spaces of jointly bounded geometry*[50]*.*

Then there exists an associative product

$$\times\colon K^u_{p_1}(X_1) \otimes K^u_{p_2}(X_2) \to K^u_{p_1+p_2}(X_1 \times X_2)$$

for $p_1, p_2 \geq -1$ with the following properties:

- *for the flip map $\tau\colon X_1 \times X_2 \to X_2 \times X_1$ and all elements $[T_1] \in K^u_{p_1}(X_1)$ and $[T_2] \in K^u_{p_2}(X_2)$ we have*

$$\tau_*[T_1 \times T_2] = (-1)^{p_1 p_2}[T_2 \times T_1],$$

- *we have for $g\colon Y \to Z$ a uniformly cobounded, proper Lipschitz map and elements $[T] \in K^u_{p_1}(X)$ and $[S] \in K^u_{p_2}(Y)$*

$$(\mathrm{id}_X \times g)_*[T \times S] = [T] \times g_*[S] \in K^u_{p_1+p_2}(X \times Z),$$

  and

- *denoting the generator of $K^u_0(\mathrm{pt}) \cong \mathbb{Z}$ by $[1]$, we have*

$$[T] \times [1] = [T] = [1] \times [T] \in K^u_*(X)$$

  for all $[T] \in K^u_*(X)$.

[49] To see this, suppose that, e.g., $T_1 = T_1' \oplus T_1''$. Then it suffices to show that $T_1' \times T_2 \oplus T_1'' \times T_2$ is aligned with T_1 and T_2, which is not hard to do.

[50] see Definition 3.19

3.4 Homotopy invariance

Let X and Y be locally compact, separable metric spaces with jointly bounded geometry and let $g_0, g_1\colon X \to Y$ be uniformly cobounded, proper and Lipschitz maps which are homotopic in the following sense: there is a uniformly cobounded, proper and Lipschitz map $G\colon X \times [0,1] \to Y$ with $G(x,0) = g_0(x)$ and $G(x,1) = g_1(x)$ for all $x \in X$.

Theorem 3.26. *If $g_0, g_1\colon X \to Y$ are homotopic in the above sense, then they induce the same maps $(g_0)_* = (g_1)_*\colon K_*^u(X) \to K_*^u(Y)$ on uniform K-homology.*

The proof of the above theorem is completely analogous to the non-uniform case and uses the external product. Furthermore, the above theorem is a special case of the following invariance of uniform K-homology under weak homotopies: given a uniform Fredholm module (H, ρ, T) over X, the push-forward of it under g_i is defined as $(H, \rho \circ g_i^*, T)$ and it is easily seen that these modules are weakly homotopic via the map G.

Definition 3.27 (Weak homotopies)**.** The family of uniform Fredholm modules (H, ρ_t, T_t) for $t \in [0,1]$ is a *weak homotopy* if:

- the family ρ_t is pointwise strong-* operator continuous, i.e., for all $f \in C_0(X)$ we get a path $\rho_t(f)$ in $\mathfrak{B}(H)$ that is continuous in the strong-* operator topology[51],
- the family T_t is continuous in the strong-* operator topology on $\mathfrak{B}(H)$, i.e., for all $v \in H$ we get norm continuous paths $T_t(v)$ and $T_t^*(v)$ in H, and
- for every $\varepsilon > 0$ and $f \in C_0(X)$ we have the following:

  For all $t \in [0,1]$ the operator $[T_t, \rho_t(f)]$ is a compact operator since (H, ρ_t, T_t) is a Fredholm module and can therefore be approximated up to ε by some finite rank operator k_t.[52] So let $\{v_i\}_{i=1,\ldots,I}$ be an orthonormal basis of the image of k_t and consider the strong-* operator neighbourhood $U(\varepsilon; v_1, \ldots, v_I)$[53] of $[T_t, \rho_t(f)]$ in $\mathfrak{B}(H)$. Now for every $[T_s, \rho_s(f)]$ in that neighbourhood we also consider its finite rank approximation k_s up to ε and an orthonormal basis $\{w_j\}_{j=1,\ldots,J}$ of its image. Then we require that $[T_t, \rho_t(f)]$ lies in the strong-* operator neighbourhood $U(\varepsilon; w_1, \ldots, w_J)$ of $[T_s, \rho_s(f)]$ in $\mathfrak{B}(H)$.[54]

  Additionally, we require the analogous property for $(T_t^2 - 1)\rho_t(f)$ and $(T_t - T_t^*)\rho_t(f)$.

---

[51] Recall that if H is a Hilbert space, then the *strong-* operator topology* on $\mathfrak{B}(H)$ is generated by the family of seminorms $p_v(T) := \|Tv\| + \|T^*v\|$ for all $v \in H$, where $T \in \mathfrak{B}(H)$.

[52] This finite rank operator k_t is not unique. Recall that every compact operator on a Hilbert space may be represented in the form $\sum_{n \geq 1} \lambda_n \langle f_n, \cdot \rangle g_n$, where the values λ_n are the singular values of the operator and $\{f_n\}$, $\{g_n\}$ are orthonormal families (but contrary to the λ_n they are not unique). Now we choose k_t to be the operator given by the same sum, but only with the λ_n satisfying $\lambda_n \geq \varepsilon$.

[53] For an operator $A \in \mathfrak{B}(H)$ we define

$$U(\varepsilon; v_1, \ldots, v_I) := \{B \in \mathfrak{B}(H) \mid \|(B - A)v_i\| + \|(B - A)^* v_i)\| < \varepsilon \text{ for all } i = 1, \ldots, I\}.$$

Note that the collection of all such sets $U(\varepsilon; \mathcal{V})$ for all $\varepsilon > 0$ and all finite collections $\mathcal{V} \subset H$ forms a neighbourhood basis of the strong-* operator topology at $A \in \mathfrak{B}(H)$.

[54] It follows that $\|k_t - k_s\|_{op} < 2\varepsilon$, which is the crucial thing that we need.

If ρ_t is pointwise norm continuous and T_t is norm continuous, then the modules are automatically weakly homotopic. So weak homotopy generalizes operator homotopy.

Remark 3.28. Since the family T_t is continuous in the strong-* operator topology and since it is defined on the compact interval $[0,1]$, we conclude with the uniform boundedness principle $\sup_t \|T_t\|_{op} < \infty$. Furthermore, we have $\|\rho_t(f)\|_{op} \leq \|f\|_\infty$ for all $t \in [0,1]$ since ρ_t are representations of C^*-algebras. Now though multiplication is not continuous as a map $\mathfrak{B}(H) \times \mathfrak{B}(H) \to \mathfrak{B}(H)$, where $\mathfrak{B}(H)$ is equipped with the strong-* operator topology, it is continuous if restricted to norm bounded subsets of $\mathfrak{B}(H)$. So all three families $[T_t, \rho_t(f)]$, $(T_t^2 - 1)\rho_t(f)$ and $(T_t - T_t^*)\rho_t(f)$ are also continuous in the strong-* operator topology.

Theorem 3.29. *Let (H, ρ_0, T_0) and (H, ρ_1, T_1) be weakly homotopic uniform Fredholm modules over a locally compact and separable metric space X of jointly bounded geometry.*

Then they define the same uniform K-homology class.

Proof. Let our weakly homotopic family (H, ρ_t, T_t) be parametrized on the interval $t \in [0, 2\pi]$, so that our notation here will coincide with the one in the proof of [Kas81, Theorem 1 in §6] that we mimic. Furthermore, we assume that ρ_t and T_t are constant in the intervals $[0, 2\pi/3]$ and $[4\pi/3, 2\pi]$

We consider the graded Hilbert space $\mathcal{H} := H \mathbin{\hat{\otimes}} (L^2[0,2\pi] \oplus L^2[0,2\pi])$ (where the space $L^2[0,2\pi] \oplus L^2[0,2\pi]$ is graded by interchanging the summands).

The family T_t maps continuous paths v_t in H again to continuous paths $T_t(v_t)$: indeed, if $t_n \to t$ is a convergent sequence, we get

$$\begin{aligned}\|T_{t_n}(v_{t_n}) - T_t(v_t)\| &\leq \|T_{t_n}(v_{t_n}) - T_{t_n}(v_t)\| + \|T_{t_n}(v_t) - T_t(v_t)\| \\ &\leq \underbrace{\|T_{t_n}\|_{op}}_{<\infty} \cdot \underbrace{\|v_{t_n} - v_t\|}_{\to 0} + \underbrace{\|(T_{t_n} - T_t)(v_t)\|}_{\to 0},\end{aligned}$$

where the second limit to 0 holds due to the continuity of T_t in the strong-* operator topology. So the family T_t maps the dense subspace $H \otimes C[0,2\pi]$ of $H \otimes L^2[0,2\pi]$ into itself, and since it is norm bounded from above by $\sup_t \|T_t\|_{op} < \infty$, it defines a bounded operator on $H \otimes L^2[0,2\pi]$. We define an odd operator $\begin{pmatrix} 0 & T_t^* \\ T_t & 0 \end{pmatrix}$ on $\mathcal{H}$, which we also denote by T_t (there should arise no confusion by using the same notation here).

Since $\rho_t(f)$ is strong-* continuous in t, we can analogously show that it maps continuous paths v_t in H again to continuous paths $\rho_t(f)(v_t)$, and it is norm bounded from above by $\|f\|_\infty$. So $\rho_t(f)$ defines a bounded operator on $H \otimes L^2[0,2\pi]$ and we can get a representation $\rho_t \oplus \rho_t$ of $C_0(X)$ on $\mathcal{H}$ by even operators, that we denote by the symbol ρ_t (again, no confusion should arise by using the same notation).

We consider now the uniform Fredholm module

$$(\mathcal{H}, \rho_t, N_1(T_t) + N_2(1 \mathbin{\hat{\otimes}} T(f)),$$

where $T(f)$ is defined in the proof of [Kas81, Theorem 1 in §6] (unfortunately, the overloading of the symbol "T" is unavoidable here). For the convenience of the reader,

we will recall the definition of the operator $T(f)$ in a moment. That we may find a suitable partition of unity N_1, N_2 is due to the last bullet point in the definition of weak homotopies, and the construction of N_1, N_2 proceeds as in the end of the proof of our Proposition 3.23.

To define $T(f)$, we first define an operator $d\colon L^2[0,2\pi] \to L^2[0,2\pi]$ using the basis $1,\dots,\cos nx,\dots,\sin nx,\dots$ by the formulas

$$d(1) := 0,\ d(\sin nx) := \cos nx \text{ and } d(\cos nx) := -\sin nx.$$

This operator d is anti-selfadjoint, $d^2+1 \in \mathfrak{K}(L^2[0,2\pi])$, and d commutes modulo compact operators with multiplication by functions from $C[0,2\pi]$. Let $f \in C[0,2\pi]$ be a continuous, real-valued function with $|f(x)| \le 1$ for all $x \in [0,2\pi]$, $f(0)=1$ and $f(2\pi)=-1$. Then we set $T_1(f) := f - \sqrt{1-f^2}\cdot d \in \mathfrak{B}(L^2[0,2\pi])$. This operator $T_1(f)$ is Fredholm and with 1, both $1 - T_1(f)\cdot T_1(f)^*$ and $1 - T_1(f)^*\cdot T_1(f)$ are compact, and $T_1(f)$ commutes modulo compacts with multiplication by functions from $C[0,2\pi]$. Furthermore, any two operators of the form $T_1(f)$ (for different f) are connected by a norm continuous homotopy consisting of operators having the same form. Finally, we define $T(f) := \begin{pmatrix} 0 & T_1(f)^* \\ T_1(f) & 0 \end{pmatrix} \in \mathfrak{B}(L^2[0,2\pi] \oplus L^2[0,2\pi])$.

We assume the our homotopies ρ_t and T_t are constant in the intervals $[0,2\pi/3]$ and $[4\pi/3,2\pi]$. Furthermore, we set

$$f(t) := \begin{cases} \cos 3t, & 0 \le t \le \pi/3, \\ -1, & \pi/3 \le t \le 2\pi. \end{cases}$$

Then $T_1(f)$ commutes with the projection P onto $L^2[0,2\pi/3]$, $P\cdot T_1(f)$ is an operator of index 1 on $L^2[0,2\pi/3]$, and $(1-P)T_1(f) \equiv -1$ on $L^2[2\pi/3,2\pi]$. We choose $\alpha(t) \in C[0,2\pi]$ with $0 \le \alpha(t) \le 1$, $\alpha(t)=0$ for $t \le \pi/3$, and $\alpha(t)=1$ for $t \ge 2\pi/3$. Using a norm continuous homotopy, we replace N_1 and N_2 by

$$\widetilde{N_1} := \sqrt{1\mathbin{\hat\otimes}(1-\alpha)}\cdot N_1 \cdot \sqrt{1\mathbin{\hat\otimes}(1-\alpha)}$$

and

$$\widetilde{N_2} := 1\mathbin{\hat\otimes}\alpha + \sqrt{1\mathbin{\hat\otimes}(1-\alpha)}\cdot N_2 \cdot \sqrt{1\mathbin{\hat\otimes}(1-\alpha)}.$$

The operator $\widetilde{N_1}(T_t) + \widetilde{N_2}(1\mathbin{\hat\otimes}T(f))$ commutes with $1\mathbin{\hat\otimes}(P\oplus P)$ and we obtain for the decomposition $L^2[0,2\pi]\oplus L^2[0,2\pi] = \operatorname{im}(P\oplus P)\oplus\operatorname{im}(1-P\oplus P)$

$$\big(\mathcal{H},\rho_t,\widetilde{N_1}(T_t)+\widetilde{N_2}(1\mathbin{\hat\otimes}T(f))\big) = \big((H,\rho_0,T_0)\times[1]\big)\oplus\big(\text{degenerate}\big),$$

where $[1] \in K_0^u(\mathrm{pt})$ is the multiplicative identity (see the third point of Theorem 3.25) and recall that we assumed that ρ_t and T_t are constant in the intervals $[0,2\pi/3]$ and $[4\pi/3,2\pi]$.

Setting

$$f(t) := \begin{cases} 1, & 0 \le t \le 5\pi/3, \\ -\cos 3t, & 5\pi/3 \le t \le 2\pi, \end{cases}$$

we get analogously

$$\big(\mathcal{H}, \rho_t, \overline{N_1}(T_t) + \overline{N_2}(1 \hat{\otimes} T(f)\big) = \big(\text{degenerate}\big) \oplus \big((H, \rho_1, T_1) \times [1]\big),$$

for suitably defined operators $\overline{N_1}$ and $\overline{N_2}$ (their definition is similar to the one of $\widetilde{N_1}$ and $\widetilde{N_2}$). Putting all the homotopies of this proof together, we get that the modules $\big((H, \rho_0, T_0) \times [1]\big) \oplus \big(\text{degenerate}\big)$ and $\big((H, \rho_1, T_1) \times [1]\big) \oplus \big(\text{degenerate}\big)$ are operator homotopic, from which the claim follows. □

3.5 Rough Baum–Connes conjecture

Špakula constructed in [Špa09, Section 9] the *rough*[55] *assembly map*

$$\mu_u \colon K_*^u(X) \to K_*(C_u^*(Y)),$$

where $Y \subset X$ is a uniformly discrete quasi-lattice, X a proper metric space, and $C_u^*(Y)$ the uniform Roe algebra of Y.[56] In this section we will discuss implications on the rough assembly map following from the properties of uniform K-homology that we have proved in the last sections.

Using homotopy invariance of uniform K-homology we will strengthen Špakula's results from [Špa09, Section 10].

Definition 3.30 (Rips complexes)**.** Let Y be a discrete metric space and let $d \geq 0$. The *Rips complex $P_d(Y)$ of Y* is a simplicial complex, where

- the vertex set of $P_d(Y)$ is Y, and
- vertices $y_0, \ldots, y_q$ span a q-simplex if and only if we have $d(y_i, y_j) \leq d$ for all $0 \leq i, j \leq q$.

Note that if Y has coarsely bounded geometry, then the Rips complex $P_d(Y)$ is uniformly locally finite and finite dimensional and therefore also, especially, a simplicial complex of bounded geometry (i.e., the number of simplices in the link of each vertex is uniformly bounded). So if we equip $P_d(Y)$ with the metric derived from barycentric coordinates, $Y \subset P_d(Y)$ becomes a quasi-lattice (cf. Examples 3.15).

Now we may state the *rough Baum–Connes conjecture*:

Conjecture 3.31. *Let Y be a proper and uniformly discrete metric space with coarsely bounded geometry.*

Then

$$\mu_u \colon \lim_{d \to \infty} K_*^u(P_d(Y)) \to K_*(C_u^*(Y))$$

is an isomorphism.

[55] We could have also called it the *uniform coarse* assembly map, but the uniform coarse category is also called the rough category and therefore we stick to this shorter name.

[56] Recall that one possible model for the uniform Roe algebra $C_u^*(Y)$ is the norm closure of the *-algebra of all finite propagation operators in $\mathfrak{B}(\ell^2(Y))$ with uniformly bounded coefficients.

Let us relate the conjecture quickly to manifolds of bounded geometry. First we need the following notion:

Definition 3.32 (Uniformly contractible spaces)**.** A metric space X is called *uniformly contractible*, if for every $r > 0$ there is an $s > 0$ such that every ball $B_r(x)$ can be contracted to a point in the ball $B_s(x)$.

The for us most important examples of uniformly contractible spaces are universal covers of aspherical Riemannian manifolds equipped with the pull-back metric.

Theorem 3.33. *Let M be a uniformly contractible manifold of bounded geometry and without boundary and let $Y \subset M$ be a uniformly discrete quasi-lattice in M.*

Then we have a natural isomorphism

$$\lim_{d\to\infty} K_*^u(P_d(Y)) \cong K_*^u(M).$$

The proof of this theorem is analogous to the corresponding non-uniform statement $\lim_{d\to\infty} K_*(P_d(Y)) \cong K_*(M)$ from [Yu95b, Theorem 3.2] and uses crucially the homotopy invariance of uniform K-homology.

Let us now relate the rough Baum–Connes conjecture to the usual Baum–Connes conjecture: let Γ be a countable, discrete group and denote by $|\Gamma|$ the metric space obtained by endowing Γ with a proper, left-invariant metric. Then $|\Gamma|$ becomes a proper, uniformly discrete metric space with coarsely bounded geometry. Note that we can always find such a metric and that any two of such metrics are quasi-isometric. If Γ is finitely generated, an example is the word metric.

Špakula proved in [Špa09, Corollary 10.3] the following equivalence of the rough Baum–Connes conjecture with the usual one: let Γ be a torsion-free, countable, discrete group. Then the rough assembly map

$$\mu_u\colon \lim_{d\to\infty} K_*^u(P_d|\Gamma|) \to K_*(C_u^*|\Gamma|)$$

is an isomorphism if and only if the Baum–Connes assembly map

$$\mu\colon K_*^\Gamma(\underline{E}\Gamma; \ell^\infty(\Gamma)) \to K_*(C_r^*(\Gamma, \ell^\infty(\Gamma)))$$

for Γ with coefficients in $\ell^\infty(\Gamma)$ is an isomorphism. For the definition of the Baum–Connes assembly map with coefficients the unfamiliar reader may consult the original paper [BCH94, Section 9]. Furthermore, the equivalence of the usual (i.e., non-uniform) coarse Baum–Connes conjecture with the Baum–Connes conjecture with coefficients in $\ell^\infty(\Gamma, \mathfrak{K})$ was proved by Yu in [Yu95a, Theorem 2.7].

Špakula mentioned in [Špa09, Remark 10.4] that the above equivalence does probably also hold without any assumptions on the torsion of Γ, but the proof of this would require some degree of homotopy invariance of uniform K-homology. So again we may utilize our proof of the homotopy invariance of uniform K-homology and therefore drop the assumption about the torsion of Γ.

Theorem 3.34. *Let Γ be a countable, discrete group.*
Then the rough assembly map

$$\mu_u \colon \lim_{d \to \infty} K_*^u(P_d|\Gamma|) \to K_*(C_u^*|\Gamma|)$$

is an isomorphism if and only if the Baum–Connes assembly map

$$\mu \colon K_*^\Gamma(\underline{E}\Gamma; \ell^\infty(\Gamma)) \to K_*(C_r^*(\Gamma, \ell^\infty(\Gamma)))$$

for Γ with coefficients in $\ell^\infty(\Gamma)$ is an isomorphism.

3.6 Homology classes of uniform elliptic operators

We will show that symmetric and elliptic pseudodifferential operators of positive order naturally define classes in uniform K-homology. This result is a crucial generalization of [Špa09, Theorem 3.1], where this statement is proved for generalized Dirac operators.

First we need a definition and then we will plunge right into the main result:

Definition 3.35 (Normalizing functions)**.** A smooth function $\chi \colon \mathbb{R} \to [-1, 1]$ is called a *normalizing function*, if

- χ is odd, i.e., $\chi(x) = -\chi(-x)$ for all $x \in \mathbb{R}$,
- $\chi(x) > 0$ for all $x > 0$ and
- $\chi(x) \to \pm 1$ for $x \to \pm\infty$.

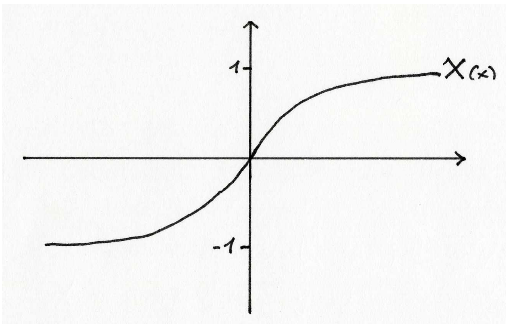

Figure 2: A normalizing function.

Theorem 3.36. *Let M be a manifold of bounded geometry and without boundary, $E \to M$ be a p-multigraded vector bundle of bounded geometry, $P \in \mathrm{U}\Psi\mathrm{DO}^k(E)$ be a symmetric and elliptic pseudodifferential operator on E of positive order $k \geq 1$, and let P be odd and multigraded.*

Then $(H, \rho, \chi(P))$ is a p-multigraded uniform Fredholm module over M, where the Hilbert space is $H := L^2(E)$, the representation $\rho \colon C_0(M) \to \mathfrak{B}(H)$ is the one via multiplication operators and χ is a normalizing function. Furthermore, the uniform K-homology class $[(H, \rho, \chi(P))] \in K_p^u(M)$ does not depend on the choice of χ.

Proof. Recall from Definition 3.6 that for the first statement that $(H, \rho, \chi(P))$ defines an ungraded uniform Fredholm module over M we have to show that $\chi(P)$ is uniformly pseudolocal and that $\chi(P)^2 - 1$ and $\chi(P) - \chi(P)^*$ are uniformly locally compact.

Since χ is real-valued and P essentially self-adjoint by Proposition 2.42, we have $\chi(P) - \chi(P)^* = 0$, i.e., the operator $\chi(P) - \chi(P)^*$ is trivially uniformly locally compact. Moreover, since we have $\chi(P)^2 - 1 = (\chi^2 - 1)(P)$ and $\chi^2 - 1 \in C_0(\mathbb{R})$, we conclude with Corollary 2.46 that $\chi(P)^2 - 1$ is uniformly locally compact.

Because the difference of two normalizing functions is a function from $C_0(\mathbb{R})$, we conclude from the same corollary that in order to show that $\chi(P)$ is uniformly pseudolocal, it suffices to show this for one particular normalizing function (and secondly, we get that the class $[(H, \rho, \chi(P))]$ is independent of the concrete choice of χ due to Lemma 3.8).

From now on we proceed as in the proof of [Špa09, Theorem 3.1] using the same formulas: we choose the particular normalizing function $\chi(x) := \frac{x}{\sqrt{1+x^2}}$ to prove that $\chi(P)$ is uniformly pseudolocal. We have $\chi(P) = \frac{2}{\pi}\int_0^\infty \frac{P}{1+\lambda^2+P^2} d\lambda$ with convergence of the integral in the strong operator topology[57] and get then for $f \in L\text{-}\operatorname{Lip}_R(M)$

$$[\rho(f), \chi(P)] = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2+P^2}\big((1+\lambda^2)[\rho(f), P] + P[\rho(f), P]P\big)\frac{1}{1+\lambda^2+P^2} d\lambda.$$

Suppose $f \in L\text{-}\operatorname{Lip}_R(M) \cap C_b^\infty(M)$. Then the integral converges in operator norm[58] and there exists an $N > 0$ which depends only on an $\varepsilon > 0$, $R = \operatorname{diam}(\operatorname{supp} f)$ and the norms of the derivatives of f,[59] such that there are $\lambda_1, \ldots, \lambda_N$ and the integral is at most ε away from the sum of the integrands for $\lambda_1, \ldots, \lambda_N$.

Now we recall Definition 2.47 of the symbol classes on $\mathbb{R}$:

$$\mathcal{S}^m(\mathbb{R}) := \{g \in C^\infty(\mathbb{R}) \mid |g^{(n)}(x)| < C_l(1+|x|)^{m-n} \text{ for all } n \in \mathbb{N}_0\}.$$

Since both $\frac{1}{1+\lambda^2+x^2} \in \mathcal{S}^{-2}(\mathbb{R})$ and $\frac{1+\lambda^2}{1+\lambda^2+x^2} \in \mathcal{S}^{-2}(\mathbb{R})$ (with respect to the variable x, i.e., for fixed λ), the operators $\frac{1}{1+\lambda^2+P^2}$ and $\frac{1+\lambda^2}{1+\lambda^2+P^2}$ are quasilocal operators of order $-2k$ by Proposition 2.48. This also holds for their adjoints and so, by Corollary 2.30, they are uniformly locally compact. The same conclusion applies to the operators $\frac{P}{1+\lambda^2+P^2}$ and $\frac{(1+\lambda^2)P}{1+\lambda^2+P^2}$ which are quasilocal of order $-k$ and hence also uniformly locally compact.

So the first summand

$$\frac{1+\lambda^2}{1+\lambda^2+P^2}[\rho(f), P]\frac{1}{1+\lambda^2+P^2}$$

[57] This follows from the equality $\frac{x}{\sqrt{1+x^2}} = \frac{2}{\pi}\int_0^\infty \frac{x}{1+\lambda^2+x^2} d\lambda$ for all $x \in \mathbb{R}$.

[58] To see this, we have to find upper bounds for the operator norms $\|\cdot\|_{0,k-1}$ of $\frac{1+\lambda^2}{1+\lambda^2+P^2}\frac{1}{1+\lambda^2+P^2}$ and $\frac{P}{1+\lambda^2+P^2}\frac{P}{1+\lambda^2+P^2}$, that are integrable with respect to λ. This can be done by, e.g., using the estimates derived in the proof of Proposition 2.48. But note that we need the generalization of this proposition to all $m \in \mathbb{R}$. For the definition of the corresponding Sobolev norms we have to use (2.2) with fixed coordinate charts, corresponding partition of unity and chosen synchronous framing. Since different choices lead to equivalent norms, our needed result that the integrand is integrable with respect to λ is independent of these choices.

[59] The dependence on R and on the derivatives of f comes from the operator norm estimate of $[\rho(f), P]$.

of the integrand is the difference of two compact operators and their approximability by finite rank operators depends only on $R = \operatorname{diam}(\operatorname{supp} f)$ and the Lipschitz constant L of f. The same also applies to the second summand

$$\frac{1}{1+\lambda^2+P^2} P[\rho(f), P] P \frac{1}{1+\lambda^2+P^2}$$

of the integrand (note that $\frac{P^2}{1+\lambda^2+P^2}$ is a bounded operator).

So the operator $[\rho(f), \chi(P)]$ is for $f \in L\text{-}\operatorname{Lip}_R(M) \cap C_b^\infty(M)$ compact and its approximability by finite rank operators depends only on R, L and the norms of the derivatives of f. That this suffices to conclude that the operator is uniformly pseudolocal is exactly Point 5 in Lemma 2.32.

To conclude the proof we have to show that $\chi(P)$ is odd and multigraded. But this was already shown in full generality in [HR00, Lemma 10.6.2]. □

We have shown in the above theorem that an elliptic pseudodifferential operator naturally defines a class in uniform K-homology. Now we will show that this class does only depend on the principal symbol of the pseudodifferential operator. Note that ellipticity of an operator does only depend on its symbol (since it is actually defined that way, see Definition 2.37, which is possible due to Lemma 2.36), i.e., another pseudodifferential operator with the same symbol is automatically also elliptic.

Proposition 3.37. *The uniform K-homology class of a symmetric and elliptic pseudodifferential operator $P \in \mathrm{U\Psi DO}^{k \geq 1}(E)$ does only depend on its principal symbol $\sigma(P)$, i.e., any other such operator P' with the same principal symbol defines the same uniform K-homology class.*

Proof. Consider in $\mathrm{U\Psi DO}^k(E)$ the linear path $P_t := (1-t)P + tP'$ of operators. They are all symmetric and, since $\sigma(P) = \sigma(P')$, they all have the same principal symbol. So they are all elliptic and therefore we get a family of uniform Fredholm modules $(H, \rho, \chi(P_t))$, where we use a fixed normalizing function χ.

Now if the family $\chi(P_t)$ of bounded operators would be norm-continuous, the claim that we get the same uniform K-homology classes would follow directly from the relations defining uniform K-homology. But it seems that in general it is only possible to conclude the norm continuity of $\chi(P_t)$ if the difference $P - P'$ is a bounded operator,[60] i.e., if the order k of P is 1 (since then the order of the difference $P - P'$ would be 0, i.e., it would define a bounded operator on $L^2(E)$).

In the case $k > 1$ we get continuity of $\chi(P_t)$ only in the strong-* operator topology on $\mathfrak{B}(L^2(E))$. This is easily seen with Proposition 2.49.[61] To conclude in this case that $(H, \rho, \chi(P_0))$ and $(H, \rho, \chi(P_1))$ define the same class in uniform K-homology, we will use Theorem 3.29, i.e., we will show now that the family $(H, \rho, \chi(P_t))$ is a weak homotopy.

The first bullet point of the definition of a weak homotopy is clearly satisfied since our representation ρ is fixed, i.e., does not depend on the time t. Moreover, we have already

[60] see, e.g., [HR00, Proposition 10.3.7]

[61] An example of a normalizing function χ fulfilling the prerequisites of Proposition 2.49 may be found in, e.g., [HR00, Exercise 10.9.3].

incidentally discussed the second bullet point in the paragraph above, so it remains to varify that the third point is satisfied. We start with investigating the operators $[\rho(f), \chi(P_t)]$. Let χ be the normalizing function $\chi(x) = \frac{x}{\sqrt{1+x^2}}$ (this is the one used in the proof of the above Theorem 3.36). It satisfies the assumptions of Proposition 2.49 since the integral $\int |s\hat{\chi}(s)| ds$ has a finite value (we will use this at the end of this paragraph). From the end of the proof of the above Theorem 3.36 we get that the approximation of $[\rho(f), \chi(P_t)]$ up to an ε via finite rank operators is done by approximating finitely many quasilocal operators of negative order times the operator $\rho(f)$. But from the proof of Proposition 2.29 (where we do this approximation), we see that we actually approximate the compact inclusions of Sobolev spaces into the L^2-space. So the images of these finite rank operators consist of functions from a Sobolev space of appropriate order and, this is the most important, the Sobolev norms of L^2-orthonormal basis of these images may be bounded from above independently of the time t, i.e., by the same bound for all operators $[\rho(f), \chi(P_t)]$. But this together with the norm estimate from Proposition 2.49 shows that the third bullet point in the Definition 3.27 of weak homotopies is fulfilled.

The arguments for $\rho(f)(\chi(P_t)^2 - 1)$ are similar and the case of $\rho(f)(\chi(P_t) - \chi(P_t)^*)$ is clear since $\chi(P_t) - \chi(P_t)^* = 0$, because P_t is essentially self-adjoint. $\square$

4 Uniform K-theory

In this section we will define uniform K-theory and show that for spin^c manifolds it is Poincaré dual to uniform K-homology. The definition of uniform K-theory is based on the following observation: We want that it consists of vector bundles such that Dirac operators over manifolds of bounded geometry may be twisted with them. So we want the vector bundles to have bounded geometry, because otherwise the twisted operator will not be uniform. Now we have $K^0_{\mathrm{cpt}}(M) \cong K_0(C_0(M))$ and we want that something similar holds in the case of vector bundles of bounded geometry that we are interested in. So first of all we have to consider functions on M which are not compactly supported since we want vector bundles that are not necessarily trivial outside some compact subset.

Then we have to guess how to incorporate the bounded geometry into this: The first try is to look at the algebra $C_b^\infty(M)$ of smooth functions on M whose derivatives are all bounded, since one can speculate that this boundedness of the derivatives translates somehow into a boundedness of the Christoffel symbols if one equips the vector bundle with the induced metric and connection coming from the given embedding of the bundle into $\mathbb{C}^k$ (this embedding is given since a projection matrix with entries in $C_b^\infty(M)$ defines a subbundle of $\mathbb{C}^k$ by considering the image of the projection matrix). The nice thing is now that this first try is successful, but unfortunately the computation of the dependence of the Christoffel symbols on the derivatives of the entries of the given projection matrix is not easily carried out. Therefore this section about uniform K-theory is quite technical and heavy in computations.

The proof of the uniform K-Poincaré duality is also unfortunately quite technical and has a lot of small detail one has to care about. So the corresponding Section 4.4 where we prove it got quite long.

Note that other authors have, of course, investigated similar versions of K-theory: Kaad investigated in [Kaa13] Hilbert bundles of bounded geometry over manifolds of bounded geometry (the author thanks Magnus Goffeng for pointing to that publication). Dropping the condition that the bundles must have bounded geometry, there is a general result by Morye contained in [Mor13] having as a corollary the Serre–Swan theorem for smooth vector bundles over (possibly non-compact) smooth manifolds. If one is only interested in the last mentioned result, there is also the short note [Sar01] by Sardanashvily. But the version of K-theory that we introduce here is, to the knowledge of the author, new.

4.1 Definition and basic properties of uniform K-theory

As we have written above, we will define uniform K-theory of a manifold of bounded geometry as the operator K-theory of $C_b^\infty(M)$. But since $C_b^\infty(M)$ turns out to be a local C^*-algebra (see Lemma 4.6), its operator K-theory will coincide with the K-theory of its closure which is the C^*-algebra $C_u(M)$ of all bounded, uniformly continuous functions on M (see Lemma 4.7). Therefore we may define uniform K-theory for any metric space X as the operator K-theory of $C_u(X)$.

Definition 4.1 (Uniform K-theory)**.** Let X be a metric space. The *uniform K-theory groups of X* are defined as

$$K_u^p(X) := K_{-p}(C_u(X)),$$

where $C_u(X)$ is the C^*-algebra of bounded, uniformly continuous functions on X.

The introduction of the minus sign in the index $-p$ in the above definition is just a convention which ensures that the indices in formulas, like the one for the cap product between uniform K-theory and uniform K-homology, coincide with the indices from the corresponding formulas for (co-)homology. Since complex K-theory is 2-periodic, the minus sign does not change anything in the formulas.

Denoting by $\overline{X}$ the completion of the metric space X, we have $K_u^*(\overline{X}) = K_u^*(X)$ because every uniformly continuous function on X has a unique extension to $\overline{X}$, i.e., $C_u(\overline{X}) = C_u(X)$. This means that, e.g., the uniform K-theories of the spaces $[0,1]$, $[0,1)$ and $(0,1)$ are all equal. Furthermore, since on a compact space X we have $C_u(X) = C(X)$, uniform K-theory coincides for compact spaces with usual K-theory. Let us state this as a small lemma:

Lemma 4.2. *If X is compact, then $K_u^*(X) = K^*(X)$.*

Remark 4.3. Note some subtle differences between uniform K-theory and uniform K-homology. Whereas uniform K-theory of X coincides with the uniform K-theory of the completion $\overline{X}$, this is in general not true for uniform K-homology.

Recall that in Proposition 3.7 we have shown that if X is totally bounded, then the uniform K-homology of X coincides with the usual K-homology of X. So for, e.g., the open unit ball in $\mathbb{R}^n$ uniform and usual K-homology coincide, and they coincide also for the closed ball. But of course the usual K-homologies of the open and closed balls are not always equal.

Contrary to this the uniform K-theory of the open ball equals the uniform K-theory of the closed ball, as we have seen in the discussion above. But we generally do not have that, as for uniform K-homology, uniform K-theory of a totally bounded space which is not compact equal usual K-theory.

Recall that in Lemma 3.17 we have shown that the uniform K-homology group $K_0^u(Y)$ of such a space is isomorphic to the group $\ell^\infty_\mathbb{Z}(Y)$ of all bounded, integer-valued sequences indexed by Y, and that $K_1^u(Y) = 0$. Since we want uniform K-theory to be dual to uniform K-homology, we need the corresponding result for uniform K-theory.

Lemma 4.4. *Let Y be a uniformly discrete metric space. Then $K_u^0(Y)$ is isomorphic to $\ell^\infty_\mathbb{Z}(Y)$ and $K_u^1(Y) = 0$.*

The proof is an easy consequence of the fact that $C_u(Y) \cong \prod_{y \in Y} C(y) \cong \prod_{y\in Y} \mathbb{C}$ for a uniformly discrete space Y, where the direct product of C^*-algebras is equipped with the pointwise algebraic operations and the sup-norm. The computation of the operator K-theory of $\prod_{y \in Y} \mathbb{C}$ is now easily done (cf. [HR00, Exercise 7.7.3]).

And last, we will give a relation of uniform K-theory with amenability. Note that an analogous relation for bounded de Rham cohomology is already well-known, and also for other, similar (co-)homology theories (see, e.g., [BW97, Section 8]).

Lemma 4.5. *Let M be a metric space with amenable fundamental group.*

We let X be the universal cover of M and we denote the covering projection by $\pi\colon X \to M$. Then the pull-back map $K_u^*(M) \to K_u^*(X)$ is injective.

Proof. The projection π induces a map $\pi^*\colon C_u(M) \to C_u(X)$ which then induces the pull-back map $K_u^*(M) \to K_u^*(X)$. We will prove the lemma by constructing a left inverse to the above map π^*, i.e., we will construct a map $p\colon C_u(X) \to C_u(M)$ with $p \circ \pi^* = \mathrm{id}\colon C_u(M) \to C_u(M)$.

Let $F \subset X$ be a fundamental domain for the action of the deck transformation group on X. Since $\pi_1(M)$ is amenable, we choose a Følner sequence $(E_i)_i \subset \pi_1(M)$ in it. Now given a function $f \in C_u(X)$, we set

$$f_i(y) := \frac{1}{\# E_i} \sum_{x \in \pi^{-1}(y) \cap E_i \cdot F} f(x)$$

for $y \in M$. This gives us a sequence of functions f_i on M, but they are in general not even continuous.

Now choosing a functional $\tau \in (\ell^\infty)^*$ associated to a free ultrafilter on $\mathbb{N}$, we define $p(f)(y) := \tau(f_i(y))$. Due to the Følner condition on $(E_i)_i$ all discontinuities that the functions f_i may have vanish in the limit under τ, and we get a bounded, uniformly continuous function $p(f)$ on M.

It is clear that p is a left inverse to π^*. $\square$

4.2 Interpretation via vector bundles

We will show now that if M is a manifold of bounded geometry then we have a description of the uniform K-theory of M via vector bundles of bounded geometry.

To show this, we first need to show that the operator K-theory of $C_u(M)$ coincides with the operator K-theory of $C_b^\infty(M)$. This is established via the following two lemmas.

Lemma 4.6. *Let M be a manifold of bounded geometry.*
Then $C_b^\infty(M)$ is a local C^*-algebra[62].

Proof. Since $C_b^\infty(M)$ is a *-subalgebra of the C^*-algebra $C_b(M)$ of bounded continuous functions on M, then norm completion of $C_b^\infty(M)$, i.e., its closure in $C_b(M)$, is surely a C^*-algebra.

So we have to show that $C_b^\infty(M)$ and all matrix algebras over it are closed under holomorphic functional calculus. Since $C_b^\infty(M)$ is naturally a Fréchet algebra with a Fréchet topology which is finer than the sup-norm topology, by [Sch92, Corollary 2.3][63] it remains to show that $C_b^\infty(M)$ itself is closed under holomorphic functional calculus.

But that $C_b^\infty(M)$ is closed under holomorphic functional calculus is easily seen using [Sch92, Lemma 1.2], which states that a unital Fréchet algebra A with a topology finer than the sup-norm topology is closed under functional calculus if and only if the inverse $a^{-1} \in \overline{A}$ of any invertible element $a \in A$ actually lies in A. $\square$

Lemma 4.7. *Let M be a manifold of bounded geometry.*
Then the sup-norm completion of $C_b^\infty(M)$ is the C^*-algebra $C_u(M)$ of bounded, uniformly continuous functions on M.

Proof. We surely have $\overline{C_b^\infty(M)} \subset C_u(M)$. To show the converse inclusion, we have to approximate a bounded, uniformly continuous function by a smooth one with bounded derivatives. This can be done by choosing a nice cover of M with subordinate partitions of unity via Lemma 2.4 and then apply in every coordinate chart the same mollifier to the uniformly continous function.

Let us elaborate a bit more on the last sentence of the above paragraph: after choosing the nice cover and cutting a function $f \in C_u(M)$ with the subordinate partition of unity $\{\varphi_i\}$, we have transported the problem to Euclidean space $\mathbb{R}^n$ and our family of functions $\varphi_i f$ is uniformly equicontinuous (this is due to the uniform continuity of f and will be crucially important at the end of this proof). Now let ψ be a mollifier on $\mathbb{R}^n$, i.e., a smooth function with $\psi \geq 0$, $\operatorname{supp} \psi \subset B_1(0)$, $\int_{\mathbb{R}^n} \psi d\lambda = 1$ and $\psi_\varepsilon := \varepsilon^{-n}\psi(\cdot/\varepsilon) \xrightarrow{\varepsilon \to 0} \delta_0$. Since convolution satisfies $D^\alpha(\varphi_i f * \psi_\varepsilon) = \varphi_i f * D^\alpha \psi_\varepsilon$, where D^α is a directional derivative on $\mathbb{R}^n$ in the directions of the multi-index α and of order $|\alpha|$, we conclude that every mollified function $\varphi_i f * \psi_\varepsilon$ is smooth with bounded derivatives. Furthermore, we know $\|\varphi_i f * D^\alpha \psi_\epsilon\|_\infty \leq \|\varphi_i f\|_\infty \cdot \|D^\alpha \psi_\varepsilon\|_1$ from which we conclude that the bounds on the

[62]That is to say, it and all matrix algebras over it are closed under holomorphic functional calculus and its completion is a C^*-algebra.

[63]The corollary states that under the condition that the topology of a Fréchet algebra A is finer than the sup-norm topology we may conclude that if A is closed under holomorphic functional calculus, then this holds also for all matrix algebras over A.

derivatives of $\varphi_i f * \psi_\varepsilon$ are uniform in i, i.e., if we glue the functions $\varphi_i f * \psi_\epsilon$ together to a function on the manifold M (note that the functions $\varphi_i f * \psi_\epsilon$ are supported in our chosen nice cover since convolution with ψ_ε enlarges the support at most by ε), we get a function $f_\varepsilon \in C_b^\infty(M)$. It remains to show that f_ε converges to f in sup-norm, which is equivalent to the statement that $\varphi_i f * \psi_\varepsilon$ converges to $\varphi_i f$ in sup-norm and uniformly in i. But we know that

$$\big|(\varphi_i f * \psi_\epsilon)(x) - (\varphi_i f)(x)\big| \le \sup_{\substack{x \in \operatorname{supp} \varphi_i f \\ y \in B_\varepsilon(0)}} \big|(\varphi_i f)(x-y) - (\varphi_i f)(x)\big|$$

from which the claim follows since the family of functions $\varphi_i f$ is uniformly equicontinuous (recall that this followed from the uniform continuity of f and this here is actually the only point in this proof where we need that property of f). □

Since $C_b^\infty(M)$ is an m-convex Fréchet algebra[64], we can also use the K-theory for m-convex Fréchet algebras as developed by Phillips in [Phi91] to define the K-theory groups of $C_b^\infty(M)$. But this produces the same groups as operator K-theory, since $C_b^\infty(M)$ is an m-convex Fréchet algebra with a finer topology than the norm topology and therefore its K-theory for m-convex Fréchet algebras coincides with its operator K-theory by [Phi91, Corollary 7.9].

We summarize this observations in the following lemma:

Lemma 4.8. *Let M be a manifold of bounded geometry.*

Then the operator K-theory of $C_u(M)$, the operator K-theory of $C_b^\infty(M)$ and Phillips K-theory for m-convex Fréchet algebras of $C_b^\infty(M)$ are all pairwise naturally isomorphic.

So we have shown $K_u^*(M) \cong K_{-*}(C_b^\infty(M))$. In order to conclude the description via vector bundles of bounded geometry, we will need to establish the correspondence between vector bundles of bounded geometry and idempotent matrices with entries in $C_b^\infty(M)$. This will be done in the next subsections.

Isomorphism classes and complements

Let M be a manifold of bounded geometry and E and F two complex vector bundles equipped with Hermitian metrics and compatible connections.

Definition 4.9 (C^∞-boundedness / C_b^∞-isomorphy of vector bundle homomorphisms)**.** We will call a vector bundle homomorphism $\varphi\colon E \to F$ *C^∞-bounded*, if with respect to synchronous framings of E and F the matrix entries of φ are bounded, as are all their derivatives, and these bounds do not depend on the chosen base points for the framings or the synchronous framings themself.

E and F will be called *C_b^∞-isomorphic*, if there exists an isomorphism $\varphi\colon E \to F$ such that both φ and φ^{-1} are C^∞-bounded. In that case we will call the map φ a C_b^∞-isomorphism. Often we will write $E \cong F$ when no confusion can arise with mistaking it with algebraic isomorphy.

[64] That is to say, a Fréchet algebra such that its topology is given by a countable family of submultiplicative seminorms.

Using the characterization of bounded geometry via the matrix transition functions from Lemma 2.5, we immediately see that if E and F are C_b^∞-isomorphic, than E is of bounded geometry if and only if F is.

It is clear that C_b^∞-isomorphy is compatible with direct sums and tensor products, i.e., if $E \cong E'$ and $F \cong F'$ then $E \oplus F \cong E' \oplus F'$ and $E \otimes F \cong E' \otimes F'$.

We will now give a useful global characterization of C_b^∞-isomorphisms if the vector bundles have bounded geometry:

Lemma 4.10. *Let E and F have bounded geometry and let $\varphi\colon E \to F$ be an isomorphism. Then φ is a C_b^∞-isomorphism if and only if*

- *φ and φ^{-1} are bounded, i.e., $\|\varphi(v)\| \le C \cdot \|v\|$ for all $v \in E$ and a fixed $C > 0$ and analogously for φ^{-1}, and*
- *$\nabla^E - \varphi^* \nabla^F$ is bounded and also all its covariant derivatives.*

Proof. For a point $p \in M$ let $B \subset M$ be a geodesic ball centered at p, $\{x_i\}$ the corresponding normal coordinates of B, and let $\{E_\alpha(y)\}$, $y \in B$, be a framing for E. Then we may write every vector field X on B as $X = X^i \frac{\partial}{\partial x_i} = (X^1, \ldots, X^n)^T$ and every section e of E as $e = e^\alpha E_\alpha = (e^1, \ldots, e^k)^T$, where we assume the Einstein summation convention and where $\cdot^T$ stands for the transpose of the vector (i.e., the vectors are actually column vectors). Furthermore, after also choosing a framing for F, φ becomes a matrix for every $y \in B$ and $\varphi(e)$ is then just the matrix multiplication $\varphi(e) = \varphi \cdot e$. Finally, $\nabla_X^E e$ is locally given by

$$\nabla_X^E e = X(e) + \Gamma^E(X) \cdot e,$$

where $X(e)$ is the column vector that we get after taking the derivative of every entry e^j of e in the direction of X and Γ^E is a matrix of 1-forms (i.e., $\Gamma^E(X)$ is then a usual matrix that we multiply with the vector e). The entries of Γ^E are called the connection 1-forms.

Since φ is an isomorphism, the pull-back connection $\varphi^* \nabla^F$ is given by

$$(\varphi^* \nabla^F)_X e = \varphi^*(\nabla_X^F (\varphi^{-1})^* e),$$

so that locally we get

$$(\varphi^* \nabla^F)_X e = \varphi^{-1} \cdot \big(X(\varphi \cdot e) + \Gamma^F(X) \cdot \varphi \cdot e \big).$$

Using the product rule we may rewrite $X(\varphi \cdot e) = X(\varphi) \cdot e + \varphi \cdot X(e)$, where $X(\varphi)$ is the application of X to every entry of φ. So at the end we get for the difference $\nabla^E - \varphi^* \nabla^F$ in local coordinates and with respect to framings of E and F

$$(\nabla^E - \varphi^* \nabla^F)_X e = \Gamma^E(X) \cdot e - \varphi^{-1} \cdot X(\varphi) \cdot e - \varphi^{-1} \cdot \Gamma^F(X) \cdot \varphi \cdot e. \tag{4.1}$$

Since E and F have bounded geometry, by Lemma 2.5 the Christoffel symbols of them with respect to synchronous framings are bounded and also all their derivatives, and

these bounds are independent of the point $p \in M$ around that we choose the normal coordinates and the framings. Assuming that φ is a C_b^∞-isomorphism, the same holds for the matrix entries of φ and φ^{-1} and we conclude with the above Equation (4.1) that the difference $\nabla^E - \varphi^*\nabla^F$ is bounded and also all its covariant derivatives (here we also need to consult the local formula for covariant derivatives of tensor fields).

Conversely, assume that φ and φ^{-1} are bounded and that the difference $\nabla^E - \varphi^*\nabla^F$ is bounded and also all its covariant derivatives. If we denote by Γ^{diff} the matrix of 1-forms given by

$$\Gamma^{\mathrm{diff}}(X) = \Gamma^E(X) - \varphi^{-1} \cdot X(\varphi) - \varphi^{-1} \cdot \Gamma^F(X) \cdot \varphi,$$

we get from Equation (4.1)

$$X(\varphi) = \varphi \cdot (\Gamma^E(X) - \Gamma^{\mathrm{diff}}(X)) - \Gamma^F(X) \cdot \varphi.$$

Since we assumed that φ is bounded, its matrix entries must be bounded. From the above equation we then conclude that also the first derivatives of these matrix entries are bounded. But now that we know that the entries and also their first derivatives are bounded, we can differentiate the above equation once more to conclude that also the second derivatives of the matrix entries of φ are bounded, on so on. This shows that φ is C^∞-bounded. At last, it remains to see that the matrix entries of φ^{-1} and also all their derivatives are bounded. But since locally φ^{-1} is the inverse matrix of φ, we just have to use Cramer's rule. $\square$

An important property of vector bundles over compact spaces is that they are always complemented, i.e., for every bundle E there is a bundle F such that $E \oplus F$ is isomorphic to the trivial bundle. Note that this fails in general for non-compact spaces. So our important task is now to show that we have an analogous proposition for vector bundles of bounded geometry, i.e., that they are always complemented (in a suitable way).

Definition 4.11 (C_b^∞-complemented vector bundles)**.** A vector bundle E will be called C_b^∞*-complemented*, if there is some vector bundle $E^\perp$ such that $E \oplus E^\perp$ is C_b^∞-isomorphic to a trivial bundle with the flat connection.

Since a bundle with a flat connection is trivially of bounded geometry, we get that $E \oplus E^\perp$ is of bounded geometry. And since a direct sum $E \oplus E^\perp$ of vector bundles is of bounded geometry if and only if both vector bundles E and $E^\perp$ are of bounded geometry, we conclude that if E is C_b^∞-complemented, then both E and its complement $E^\perp$ are of bounded geometry. It is also clear that if E is C_b^∞-complemented and $F \cong E$, then F is also C_b^∞-complemented.

We will now prove the crucial fact that every vector bundle of bounded geometry is C_b^∞-complemented. The proof is just the usual one for vector bundles over compact Hausdorff spaces, but we additionally have to take care of the needed uniform estimates. As a source for this usual proof the author used [Hat09, Proposition 1.4].

Proposition 4.12. *Let M be a manifold of bounded geometry and let $E \to M$ be a vector bundle of bounded geometry.*

Then E is C_b^∞-complemented.

Proof. Since M and E have bounded geometry, we can find a uniformly locally finite cover of M by normal coordinate balls of a fixed radius together with a subordinate partition of unity whose derivatives are all uniformly bounded and such that over each coordinate ball E is trivialized via a synchronous framing. This follows basically from Lemma 2.4.

Now we color the coordinate balls with finitely many colors so that no two balls with the same color do intersect.[65] This gives a partition of the coordinate balls into N families $U_1, \ldots, U_N$ such that every U_i is a collection of disjoint balls, and we get a corresponding subordinate partition of unity $1 = \varphi_1 + \ldots + \varphi_N$ with uniformly bounded derivatives (each φ_i is the sum of all the partition of unity functions of the coordinate balls of U_i). Furthermore, E is trivial over each U_i and we denote these trivializations coming from the synchronous framings by $h_i \colon p^{-1}(U_i) \to U_i \times \mathbb{C}^k$, where $p \colon E \to M$ is the projection.

Now we set

$$g_i \colon E \to \mathbb{C}^k,\ g_i(v) := \varphi_i(p(v)) \cdot \pi_i(h_i(v)),$$

where $\pi_i \colon U_i \times \mathbb{C}^k \to \mathbb{C}^k$ is the projection. Each g_i is a linear injection on each fiber over $\varphi_i^{-1}(0,1]$ and so, if we define

$$g \colon E \to \mathbb{C}^{Nk},\ g(v) := (g_1(v), \ldots, g_N(v)),$$

we get a map g that is a linear injection on each fiber of E. Finally, we define a map

$$G \colon E \to M \times \mathbb{C}^{Nk},\ G(v) := (p(v), g(v)).$$

This establishes E as a subbundle of a trivial bundle.

If we equip $M \times \mathbb{C}^{Nk}$ with a constant metric and the flat connection, we get that the induced metric and connection on E is C_b^∞-isomorphic to the original metric and connection on E (this is due to our choice of G). Now let us denote by e the projection matrix of the trivial bundle $\mathbb{C}^{Nk}$ onto the subbundle $G(E)$ of it, i.e., e is an $Nk \times Nk$-matrix with functions on M as entries and $\operatorname{im} e = E$. Now, again due to our choice of G, we can conclude that these entries of e are bounded functions with all derivatives of them also bounded, i.e., $e \in \operatorname{Idem}_{Nk \times Nk}(C_b^\infty(M))$. Now the claim follows with the Proposition 4.14 which establishes the orthogonal complement $E^\perp$ of E in $\mathbb{C}^{Nk}$ with the induced metric and connection as a C_b^∞-complement to E. □

[65]Construct a graph whose vertices are the coordinate balls and two vertices are connected by an edge if the corresponding coordinate balls do intersect. We have to find a coloring of this graph with only finitely many colors (where of course connected vertices do have different colors). To do this, we firstly use the theorem of de Bruijin–Erdös stating that an infinite graph may be colored by k colors if and only if every of its finite subgraphs may be colored by k colors. Secondly, since the coordinate balls have a fixed radius and since our manifold has bounded geometry, the number of balls intersecting a fixed one is uniformly bounded from above. It follows that the number of edges attached to each vertex in our graph is uniformly bounded from above, i.e., the maximum vertex degree of our graph is finite. But this also holds for every subgraph of our graph, with the maximum vertex degree possibly only decreasing by passing to a subgraph. Now a simple greedy algorithm shows that every finite graph may be colored with one more color than its maximum vertex degree.

We have seen in the above proposition that every vector bundle of bounded geometry is C_b^∞-complemented. Now if we have a manifold of bounded geometry M, then its tangent bundle TM is of bounded geometry and so we know that it is C_b^∞-complemented (although TM is real and not a complex bundle, the above proof of course also holds for real vector bundles). But in this case we usually want the complement bundle to be given by the normal bundle NM coming from an embedding $M \hookrightarrow \mathbb{R}^N$. We will prove this now under the assumption that the embedding of M into $\mathbb{R}^N$ is "nice":[66]

Corollary 4.13. *Let M be a manifold of bounded geometry and let it be isometrically embedded into $\mathbb{R}^N$ such that the second fundamental form is C^∞-bounded.*

Then its tangent bundle TM is C_b^∞-complemented by the normal bundle NM corresponding to this embedding $M \hookrightarrow \mathbb{R}^N$, equipped with the induced metric and connection.

Proof. Let M be isometrically embedded in $\mathbb{R}^N$. Then its tangent bundle TM is a subbundle of $T\mathbb{R}^N$ and we denote the projection onto it by $\pi\colon T\mathbb{R}^N \to TM$. Because of Point 1 of the following Proposition 4.14 it suffices to show that the entries of π are C^∞-bounded functions.

Let $\{v_i\}$ be the standard basis of $\mathbb{R}^N$ and let $\{E_\alpha(y)\}$ be the orthonormal frame of TM arising out of normal coordinates $\{\partial_k\}$ of M via the Gram-Schmidt process. Then the entries of the projection matrix π with respect to the basis $\{v_i\}$ are given by

$$\pi_{ij}(y) = \sum_\alpha \langle E_\alpha(y), v_j\rangle \langle E_\alpha(y), v_i\rangle.$$

Let $\widetilde{\nabla}$ denote the flat connection on $\mathbb{R}^N$. Since $\widetilde{\nabla}_{\partial_k} v_i = 0$ we get

$$\partial_k \pi_{ij}(y) = \sum_\alpha \langle \widetilde{\nabla}_{\partial_k} E_\alpha(y), v_j\rangle \langle E_\alpha(y), v_i\rangle + \langle E_\alpha(y), v_j\rangle \langle \widetilde{\nabla}_{\partial_k} E_\alpha(y), v_i\rangle.$$

Now if we denote by ∇^M the connection on M, we get

$$\widetilde{\nabla}_{\partial_k} E_\alpha(y) = \nabla^M_{\partial_k} E_\alpha(y) + \mathrm{II}(\partial_k, E_\alpha),$$

where II is the second fundamental form. So to show that π_{ij} is C^∞-bounded, we must show that $E_\alpha(y)$ are C^∞-bounded sections of TM (since by assumption the second fundamental form is a C^∞-bounded tensor field). But that these $E_\alpha(y)$ are C^∞-bounded sections of TM follows from their construction (i.e., applying Gram-Schmidt to the normal coordinate fields ∂_k) and because M has bounded geometry. $\square$

Interpretation of $K_u^0(M)$

Recall for the understanding of the following proposition that if a vector bundle is C_b^∞-complemented, then it is of bounded geometry. Furthermore, this proposition is the crucial one that gives us the description of uniform K-theory via vector bundles of bounded geometry.

[66] See [Pet11] for a discussion of existence of "nice" embeddings.

Proposition 4.14. *Let M be a manifold of bounded geometry.*

1. *Let $e \in \mathrm{Idem}_{N\times N}(C_b^\infty(M))$ be an idempotent matrix.*

   Then the vector bundle $E := \operatorname{im} e$, equipped with the induced metric and connection, is C_b^∞-complemented.

2. *Let E be a C_b^∞-complemented vector bundle, i.e., there is a vector bundle $E^\perp$ such that $E \oplus E^\perp$ is C_b^∞-isomorphic to the trivial N-dimensional bundle $\mathbb{C}^N \to M$.*

   Then all entries of the projection matrix e onto the subspace $E \oplus 0 \subset \mathbb{C}^N$ with respect to a global synchronous framing of $\mathbb{C}^N$ are C^∞-bounded, i.e., we have $e \in \mathrm{Idem}_{N\times N}(C_b^\infty(M))$.

Proof of point 1. We denote by E the vector bundle $E := \operatorname{im} e$ and by $E^\perp$ its complement $E^\perp := \operatorname{im}(1-e)$ and equip them with the induced metric and connection. So we have to show that $E \oplus E^\perp$ is C_b^∞-isomorphic to the trivial bundle $\mathbb{C}^N \to M$.

Let $\varphi\colon E \oplus E^\perp \to \mathbb{C}^N$ be the canonical algebraic isomorphism $\varphi(v,w) := v + w$. We have to show that both φ and φ^{-1} are C^∞-bounded.

Let $p \in M$. Let $\{E_\alpha\}$ be an orthonormal basis of the vector space E_p and $\{E_\beta^\perp\}$ an orthonormal basis of $E_p^\perp$. Then the set $\{E_\alpha, E_\beta^\perp\}$ is an orthonormal basis for $\mathbb{C}_p^N$. We extend $\{E_\alpha\}$ to a synchronous framing $\{E_\alpha(y)\}$ of E and $\{E_\beta^\perp\}$ to a synchronous framing $\{E_\beta^\perp(y)\}$ of $E^\perp$. Since $\mathbb{C}^N$ is equipped with the flat connection, the set $\{E_\alpha, E_\beta^\perp\}$ forms a synchronous framing for $\mathbb{C}^N$ at all points of the normal coordinate chart. Then $\varphi(y)$ is the change-of-basis matrix from the basis $\{E_\alpha(y), E_\beta^\perp(y)\}$ to the basis $\{E_\alpha, E_\beta^\perp\}$ and vice versa for $\varphi^{-1}(y)$; see Figure 3:

We have $e(p)(E_\alpha) = E_\alpha$. Since the entries of e are C^∞-bounded and the rank of a matrix is a lower semi-continuous function of the entries, there is some geodesic ball B around p such that $\{e(y)(E_\alpha)\}$ forms a basis of E_y for all $y \in B$ and the diameter of the ball B is bounded from below independently of $p \in M$. We denote by $\Gamma_{i\nu}^\mu(y)$ the Christoffel symbols of E with respect to the frame $\{e(y)(E_\alpha)\}$. Let $\gamma(t)$ be a radial geodesic in M with $\gamma(0) = p$. If we now let $E_\alpha(\gamma(t))^\mu$ denote the μth entry of the vector $E_\alpha(\gamma(t))$ represented in the basis $\{e(\gamma(t))(E_\alpha)\}$, then (since it is a synchronous frame) it satisfies the ODE

$$\tfrac{d}{dt} E_\alpha(\gamma(t))^\mu = -\sum_{i,\nu} E_\alpha(\gamma(t))^\nu \cdot \tfrac{d}{dt}\gamma_i(t) \cdot \Gamma_{i\nu}^\mu(\gamma(t)),$$

where $\{\gamma_i\}$ is the coordinate representation of γ in normal coordinates $\{x_i\}$. Since γ is a radial geodesic, its representation in normal coordinates is $\gamma_i(t) = t \cdot \gamma_i(0)$ and so the above formula simplifies to

$$\tfrac{d}{dt} E_\alpha(\gamma(t))^\mu = -\sum_{i,\nu} E_\alpha(\gamma(t))^\nu \cdot \gamma_i(0) \cdot \Gamma_{i\nu}^\mu(\gamma(t)). \tag{4.2}$$

Since $\Gamma_{i\nu}^\mu(y)$ are the Christoffel symbols with respect to the frame $\{e(y)(E_\alpha)\}$, we get the equation

$$\sum_\mu \Gamma_{i\nu}^\mu(y) \cdot e(y)(E_\mu) = \nabla_{\partial_i}^E e(y)(E_\nu). \tag{4.3}$$

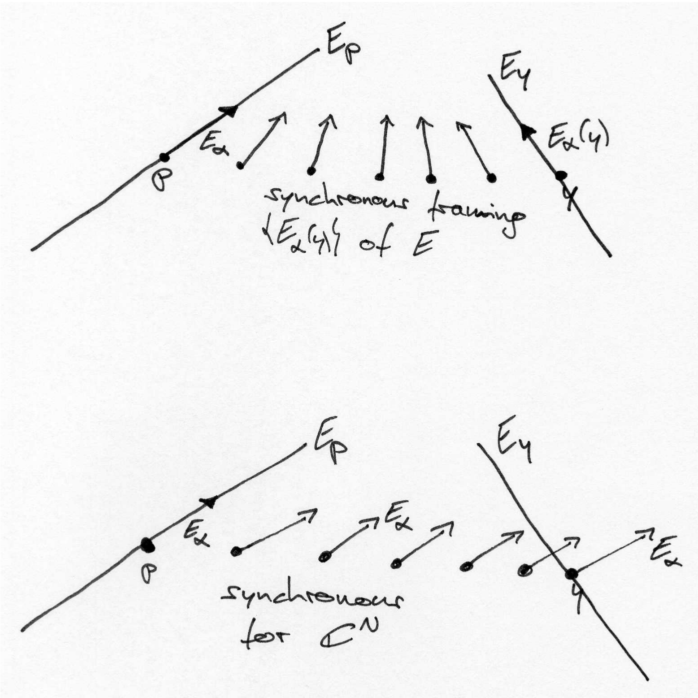

Figure 3: The frames $\{E_\alpha(y)\}$ and $\{E_\alpha\}$.

Now using that ∇^E is induced by the flat connection, we get

$$\nabla^E_{\partial_i} e(y)(E_\nu) = e(\partial_i(e(y)(E_\nu))) = e((\partial_i e)(y)(E_\nu)),$$

i.e., $e((\partial_i e)(y)(E_\nu))$ is the representation of $\nabla^E_{\partial_i} e(y)(E_\nu)$ with respect to the frame $\{E_\alpha, E^\perp_\beta\}$. Since the entries of e are C^∞-bounded, the entries of this representation $e((\partial_i e)(y)(E_\nu))$ are also C^∞-bounded. From Equation (4.3) we see that $\Gamma^\mu_{i\nu}(y)$ is the representation of $\nabla^E_{\partial_i} e(y)(E_\nu)$ in the frame $\{e(y)(E_\mu)\}$. So we conclude that the Christoffel symbols $\Gamma^\mu_{i\nu}(y)$ are C^∞-bounded functions.

Equation (4.2) and the theory of ODEs now tell us that the functions $E_\alpha(y)^\mu$ are C^∞-bounded. Since these are the representations of the vectors $E_\alpha(y)$ in the basis $\{e(y)(E_\alpha)\}$, we can conclude that the entries of the representations of the vectors $E_\alpha(y)$ in the basis $\{E_\alpha, E^\perp_\beta\}$ are C^∞-bounded. But now these entries are exactly the first $(\dim E)$ columns of the change-of-basis matrix $\varphi(y)$.

Arguing analogously for the complement $E^\perp$, we get that the other columns of $\varphi(y)$ are also C^∞-bounded, i.e., φ itself is C^∞-bounded.

It remains to show that the inverse homomorphism φ^{-1} is C^∞-bounded. But since pointwise it is given by the inverse matrix, i.e., $\varphi^{-1}(y) = \varphi(y)^{-1}$, this claim follows immediately from Cramer's rule, because we already know that φ is C^∞-bounded. $\square$

Proof of point 2. Let $\{E_\alpha(y)\}$ be a synchronous framing for E and $\{E^\perp_\beta(y)\}$ one for $E^\perp$. Then $\{E_\alpha(y), E^\perp_\beta(y)\}$ is one for $E \oplus E^\perp$. Furthermore, let $\{v_i(y)\}$ be a synchronous framing for the trivial bundle $\mathbb{C}^N$ and let $\varphi\colon E \oplus E^\perp \to \mathbb{C}^N$ be the C^∞_b-isomorphism.

Then projection matrix $e \in \mathrm{Idem}_{N\times N}(C^\infty(M))$ onto the subspace $E \oplus 0$ is given with respect to the basis $\{E_\alpha(y), E_\beta^\perp(y)\}$ of $E \oplus E^\perp$ and of $\mathbb{C}^N$ by the usual projection matrix onto the first $(\dim E)$ vectors, i.e., its entries are clearly C^∞-bounded since they are constant. Now changing the basis to $\{v_i(y)\}$, the representation of $e(y)$ with respect to this new basis is given by $\varphi^{-1}(y) \cdot e \cdot \varphi(y)$, i.e., $e \in \mathrm{Idem}_{N\times N}(C_b^\infty(M))$. □

If we have a C_b^∞-complemented vector bundle E, then different choices of complements and different choices of isomorphisms with the trivial bundle lead to similar projection matrices. The proof of this is analogous to the corresponding proof in the usual case of vector bundles over compact Hausdorff spaces. We also get that C_b^∞-isomorphic vector bundles produce similar projection matrices. Of course this also works the other way round, i.e., similar idempotent matrices give us C_b^∞-isomorphic vector bundles. Again, the proof of this is the same as the one in the topological category.

Definition 4.15. Let M be a manifold of bounded geometry. We define

- $\mathrm{Vect}_u(M)/_\sim$ as the abelian monoid of all complex vector bundles of bounded geometry over M modulo C_b^∞-isomorphism (the addition is given by the direct sum $[E] + [F] := [E \oplus F]$) and
- $\mathrm{Idem}(C_b^\infty(M))/_\sim$ as the abelian monoid of idempotent matrizes of arbitrary size over the Fréchet algebra $C_b^\infty(M)$ modulo similarity (with addition defined as $[e] + [f] := \left[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right]$).

Let $f\colon M \to N$ be a C^∞-bounded map[67] and E a vector bundle of bounded geometry over N. Then it is clear that the pullback bundle f^*E equipped with the pullback metric and connection is a vector bundle of bounded geometry over M.

The above discussion together with Proposition 4.14 prove the following corollary:

Corollary 4.16. *The monoids $\mathrm{Vect}_u(M)/_\sim$ and $\mathrm{Idem}(C_b^\infty(M))/_\sim$ are isomorphic and this isomorphism is natural with respect to C^∞-bounded maps between manifolds.*

From this Corollary 4.16, Lemma 4.6 and Proposition 4.12 we immediately get the following interpretation of the 0th uniform K-theory group $K_u^0(M)$ of a manifold of bounded geometry:

Theorem 4.17 (Interpretation of $K_u^0(M)$). *Let M be a Riemannian manifold of bounded geometry and without boundary.*

Then every element of $K_u^0(M)$ is of the form $[E] - [F]$, where both $[E]$ and $[F]$ are C_b^∞-isomorphism classes of complex vector bundles of bounded geometry over M.

Moreover, every complex vector bundle of bounded geometry defines a class in $K_u^0(M)$.

Note that the last statement in the above theorem is not trivial since it relies on the Proposition 4.12.

[67]We use covers of M and N via normal coordinate charts of a fixed radius and demand that locally in this charts the derivatives of f are all bounded and these bounds are independent of the chart used.

Interpretation of $K_u^1(M)$

For the interpretation of $K_u^1(M)$ we will make use of suspensions of algebras. The suspension isomorphism theorem for operator K-theory states that we have an isomorphism $K_1(C_u(M)) \cong K_0(SC_u(M))$, where $SC_u(M)$ is the suspension of $C_u(M)$:

$$\begin{aligned} SC_u(M) &:= \{f \colon S^1 \to C_u(M) \mid f \text{ continuous and } f(1) = 0\} \\ &\cong \{f \in C_u(S^1 \times M) \mid f(1,x) = 0 \text{ for all } x \in M\}. \end{aligned}$$

Equipped with the sup-norm this is again a C^*-algebra. Since functions $f \in SC_u(M)$ are uniformly continuous, the condition $f(1,x) = 0$ for all $x \in M$ is equivalent to $\lim_{t\to 1} f(t,x) = 0$ uniformly in x.

Now in order to interpret $K_0(SC_u(M))$ via vector bundles of bounded geometry over $S^1 \times M$, we will need to find a suitable Fréchet subalgebra of $SC_u(M)$ so that we can again use Proposition 4.14. Luckily, this was already done by Phillips in [Phi91]:

Definition 4.18 (Smooth suspension of a Fréchet algebras, [Phi91, Definition 4.7])**.** Let A be a Fréchet algebra. Then the *smooth suspension* $S_\infty A$ *of* A is defined as the Fréchet algebra

$$S_\infty A := \{f \colon S^1 \to A \mid f \text{ smooth and } f(1) = 0\}$$

equipped with the topology of uniform convergence of every derivative in every seminorm of A.

For a manifold M we have

$$\begin{aligned} S_\infty C_b^\infty(M) &\cong \{f \in C_b^\infty(S^1 \times M) \mid f(1,x) = 0 \text{ for all } x \in M\} \\ &= \{f \in C_b^\infty(S^1 \times M) \mid \forall k \in \mathbb{N}_0 \colon \lim_{t\to 1} \nabla_x^k f(t,x) = 0 \text{ uniformly in } x\}. \end{aligned}$$

The proof of the following lemma is analogous to the proof of the Lemma 4.6:

Lemma 4.19. *Let M be a manifold of bounded geometry.*

Then the sup-norm completion of $S_\infty C_b^\infty(M)$ is $SC_u(M)$ and $S_\infty C_b^\infty(M)$ is a local C^*-algebra.

Putting it all together, we get $K_u^1(M) = K_0(S_\infty C_b^\infty(M))$, and Proposition 4.14, adapted to our case here, gives us the following interpretation of the 1st uniform K-theory group $K_u^1(M)$ of a manifold of bounded geometry:

Theorem 4.20 (Interpretation of $K_u^1(M)$)**.** *Let M be a Riemannian manifold of bounded geometry and without boundary.*

Then every elements of $K_u^1(M)$ is of the form $[E] - [F]$, where both $[E]$ and $[F]$ are C_b^∞-isomorphism classes of complex vector bundles of bounded geometry over $S^1 \times M$ with the following property: there is some neighbourhood $U \subset S^1$ of 1 such that $[E|_{U\times M}]$ and $[F|_{U\times M}]$ are C_b^∞-isomorphic to a trivial vector bundle with the flat connection (the dimension of the trivial bundle is the same for both $[E|_{U\times M}]$ and $[F|_{U\times M}]$).

Moreover, every pair of complex vector bundles E and F of bounded geometry and with the above properties define a class $[E] - [F]$ in $K_u^1(M)$.

Note that the last statement in the above theorem is not trivial since it relies on the Proposition 4.12.

4.3 Cap product

In this section we will define the cap product $\cap\colon K_u^p(X) \otimes K_q^u(X) \to K_{q-p}^u(X)$ for a locally compact and separable metric space X of jointly bounded geometry[68].

Recall that we have

$$L\text{-}\operatorname{Lip}_R(X) := \{f \in C_c(X) \mid f \text{ is } L\text{-Lipschitz}, \operatorname{diam}(\operatorname{supp} f) \leq R \text{ and } \|f\|_\infty \leq 1\}.$$

Let us first describe the cap product of $K_u^0(X)$ with $K_*^u(X)$ on the level of uniform Fredholm modules. The general definition of it will be given via dual algebras.

Lemma 4.21. *Let P be a projection in $\operatorname{Mat}_{n\times n}(C_u(X))$ and let (H, ρ, T) be a uniform Fredholm module.*

We set $H_n := H \otimes \mathbb{C}^n$, $\rho_n(\cdot) := \rho(\cdot) \otimes \operatorname{id}_{\mathbb{C}^n}$, $T_n := T \otimes \operatorname{id}_{\mathbb{C}^n}$ and by π we denote the matrix $\pi_{ij} := \rho(P_{ij}) \in \operatorname{Mat}_{n\times n}(\mathfrak{B}(H)) = \mathfrak{B}(H_n)$.

Then $(\pi H_n, \pi\rho_n\pi, \pi T_n \pi)$ is a uniform Fredholm module, with an induced (multi-)grading if (H, ρ, T) was (multi-)graded.

Proof. Let us first show that the operator $\pi T_n \pi$ is a uniformly pseudolocal one. Let $R, L > 0$ be given and we have to show that $\{[\pi T_n \pi, \pi\rho_n(f)\pi] \mid f \in L\text{-}\operatorname{Lip}_R(X)\}$ is uniformly approximable. This means that we must show that for every $\varepsilon > 0$ there exists an $N > 0$ such that for every $[\pi T_n\pi, \pi\rho_n(f)\pi]$ with $f \in L\text{-}\operatorname{Lip}_R(X)$ there is a rank-N operator k with $\|[\pi T_n \pi, \pi\rho_n(f)\pi] - k\| < \varepsilon$.

We have

$$[\pi T_n \pi, \pi\rho_n(f)\pi] = \pi[T_n, \pi\rho_n(f)]\pi,$$

because $\pi^2 = \pi$ and π commutes with $\rho_n(f)$. So since $(\pi\rho_n(f))_{ij} = \rho(P_{ij}f) \in \mathfrak{B}(H)$, we get for the matrix entries of the commutator

$$([T_n, \pi\rho_n(f)])_{ij} = [T, \rho(P_{ij}f)].$$

Since the P_{ij} are bounded and uniformly continuous, they can be uniformly approximated by Lipschitz functions, i.e., there are P_{ij}^ε with

$$\|P_{ij} - P_{ij}^\varepsilon\|_\infty < \varepsilon/(4n^2\|T\|).$$

Note that we have $P_{ij}^\varepsilon f \in L_{ij}\text{-}\operatorname{Lip}_R(X)$, where L_{ij} depends only on L and P_{ij}^ε. We define $L' := \max\{L_{ij}\}$.

Now we apply the uniform pseudolocality of T, i.e., we get a maximum rank N' corresponding to R, L' and $\varepsilon/2n^2$. So let k_{ij}^ε be the rank-N' operators corresponding to the functions $P_{ij}^\varepsilon f$, i.e.,

$$\|[T, \rho(P_{ij}^\varepsilon f)] - k_{ij}^\varepsilon\| < \varepsilon/2n^2.$$

[68] see Definition 3.19

We set $k := \pi(k_{ij}^\varepsilon)\pi$, where (k_{ij}^ε) is viewed as a matrix of operators. Then k has rank at most $N := n^2 N'$. Then we compute

$$\begin{aligned}
&\|[\pi T_n \pi, \pi\rho_n(f)\pi] - k\| \\
&\quad = \|\pi[T_n, \pi\rho_n(f)]\pi - \pi(k_{ij}^\varepsilon)\pi\| \\
&\quad \leq \|\pi\|^2 \cdot n^2 \cdot \max_{i,j}\{\|[T, \rho(P_{ij}f)] - k_{ij}^\varepsilon\|\} \\
&\quad \leq \|\pi\|^2 \cdot n^2 \cdot \max_{i,j}\{\underbrace{\|[T, \rho(P_{ij}f)] - [T, \rho(P_{ij}^\varepsilon f)]\|}_{=\|[T,\rho(P_{ij}-P_{ij}^\varepsilon)\rho(f)]\|} + \underbrace{\|[T, \rho(P_{ij}^\varepsilon f)] - k_{ij}^\varepsilon\|}_{\leq \varepsilon/2n^2}\} \\
&\quad \leq \|\pi\|^2 \cdot n^2 \cdot \max_{i,j}\{2\|T\| \cdot \underbrace{\|\rho(P_{ij} - P_{ij}^\varepsilon)\| \cdot \|\rho(f)\|}_{\leq \varepsilon/(4n^2\|T\|)} + \varepsilon/2n^2\} \\
&\quad \leq \|\pi\|^2 \cdot \varepsilon,
\end{aligned}$$

which concludes the proof of the uniform pseudolocality of $\pi T_n \pi$.

That $(\pi T_n \pi)^2 - 1$ and $\pi T_n \pi - (\pi T_n \pi)^*$ are uniformly locally compact can be shown analogously. Note that because T is uniformly pseudolocal we may interchange the order of the operators T_n and $\rho(P_{ij}^\varepsilon f)$ in formulas (since for fixed R and L the subset $\{[T_n, \rho(P_{ij}^\varepsilon f)] \mid f \in L\text{-}\operatorname{Lip}_R(X)\} \subset \mathfrak{B}(H_n)$ is uniformly approximable).

We have shown that $(\pi H_n, \pi\rho_n\pi, \pi T_n \pi)$ is a uniform Fredholm module. That it inherits a (multi-)grading from (H, ρ, T) is clear and this completes the proof. □

That the construction from the above lemma is compatible with the relations defining K-theory and uniform K-homology and that it is bilinear is quickly deduced and completely analogous to the non-uniform case. So we get a well-defined pairing

$$\cap\colon K_u^0(X) \otimes K_*^u(X) \to K_*^u(X)$$

which exhibits $K_*^u(X)$ as a module over the ring $K_u^0(X)$.[69]

To define the cap product in its general form, we will use the dual algebra picture of uniform K-homology, i.e., Paschke duality:

Definition 4.22 ([Špa09, Definition 4.1])**.** Let H be a separable Hilbert space and $\rho\colon C_0(X) \to \mathfrak{B}(H)$ a representation.

We denote by $\mathfrak{D}_{\rho\oplus 0}^u(X) \subset \mathfrak{B}(H \oplus H)$ the C^*-algebra of all uniformly pseudolocal operators with respect to the representation $\rho \oplus 0$ of $C_0(X)$ on the space $H \oplus H$ and by $\mathfrak{C}_{\rho\oplus 0}^u(X) \subset \mathfrak{B}(H \oplus H)$ the C^*-algebra of all uniformly locally compact operators.

That the algebras $\mathfrak{D}_{\rho\oplus 0}^u(X)$ and $\mathfrak{C}_{\rho\oplus 0}^u(X)$ are indeed C^*-algebras was shown by Špakula in [Špa09, Lemma 4.2]. There it was also shown that $\mathfrak{C}_{\rho\oplus 0}^u(X) \subset \mathfrak{D}_{\rho\oplus 0}^u(X)$ is a closed, two-sided *-ideal.

[69] Compatibility with the internal product on $K_u^0(X)$, i.e., $(P \otimes Q) \cap T = P \cap (Q \cap T)$, is easily deduced. It mainly uses the fact that the isomorphism $\operatorname{Mat}_{n\times n}(\mathbb{C}) \otimes \operatorname{Mat}_{m\times m}(\mathbb{C}) \cong \operatorname{Mat}_{nm\times nm}(\mathbb{C})$ is canonical up to the ordering of basis elements. But different choices of orderings result in isomorphisms that differ by inner automorphisms, which makes no difference at the level of K-theory.

Definition 4.23. The groups $K^u_{-1}(X; \rho \oplus 0)$ are analogously defined as $K^u_{-1}(X)$, except that we consider only uniform Fredholm modules whose Hilbert spaces and representations are (finite or countably infinite) direct sums of $H \oplus H$ and $\rho \oplus 0$.

For $K^u_0(X; \rho \oplus 0)$ we consider only uniform Fredholm modules modeled on $H' \oplus H'$ with the representation $\rho' \oplus \rho'$, where H' is a finite or countably infinite direct sum of $H \oplus H$ and ρ' analogously a direct sum of finitely or infinitely many $\rho \oplus 0$, and the grading is given by interchanging the two summands in $H' \oplus H'$. Such Fredholm modules are called *balanced* in [HR00, Definition 8.3.10].

Proposition 4.24 ([Špa09, Proposition 4.3])**.** *The maps*

$$\varphi_* \colon K_{1+*}(\mathfrak{D}^u_{\rho\oplus 0}(X)) \to K^u_*(X; \rho \oplus 0)$$

for $* = -1, 0$ *are isomorphisms.*

Combining the above proposition with the following uniform version of Voiculescu's Theorem, we get the needed uniform version of Paschke duality.

Theorem 4.25 ([Špa10, Corollary 3.6])**.** *Let X be a locally compact and separable metric space of jointly bounded geometry and $\rho \colon C_0(X) \to \mathfrak{B}(H)$ an ample representation, i.e., ρ is non-degenerate and $\rho(f) \in \mathfrak{K}(H)$ implies $f \equiv 0$.*

Then we have

$$K^u_*(X; \rho \oplus 0) \cong K^u_*(X)$$

for both $* = -1, 0$.

The following lemma is a uniform analog of the fact [HR00, Lemma 5.4.1] and is essentially proven in [Špa09, Lemma 5.3] (by "setting $Z := \emptyset$" in that lemma).

Lemma 4.26. *We have*

$$K_*(\mathfrak{C}^u_{\rho\oplus 0}(X)) = 0$$

and so the quotient map $\mathfrak{D}^u_{\rho\oplus 0}(X) \to \mathfrak{D}^u_{\rho\oplus 0}(X)/\mathfrak{C}^u_{\rho\oplus 0}(X)$ *induces an isomorphism*

$$K_*(\mathfrak{D}^u_{\rho\oplus 0}(X)) \cong K_*(\mathfrak{D}^u_{\rho\oplus 0}(X)/\mathfrak{C}^u_{\rho\oplus 0}(X)) \tag{4.4}$$

due to the 6*-term exact sequence for K-theory.*

The last ingredient to construct the cap product is the inclusion

$$[C_u(X), \mathfrak{D}^u_{\rho\oplus 0}(X)] \subset \mathfrak{C}^u_{\rho\oplus 0}(X). \tag{4.5}$$

It is proven in the following way: let $\varphi \in C_u(X)$ and $T \in \mathfrak{D}^u_{\rho\oplus 0}(X)$. We have to show that $[\varphi, T] \in \mathfrak{C}^u_{\rho\oplus 0}(X)$. By approximating φ uniformly by Lipschitz functions we may without loss of generality assume that φ itself is already Lipschitz. Now the claim follows immediately from $f[\varphi, T] = [f\varphi, T] - [f, T]\varphi$ since T is uniformly pseudolocal.

Now we are able to define the cap product. Consider the map

$$\sigma \colon C_u(X) \otimes \mathfrak{D}^u_{\rho\oplus 0}(X) \to \mathfrak{D}^u_{\rho\oplus 0}(X)/\mathfrak{C}^u_{\rho\oplus 0}(X),\ f \otimes T \mapsto [fT].$$

It is a multiplicative *-homomorphism due to the above Equation (4.5) and hence induces a map on K-theory

$$\sigma_*\colon K_*(C_u(X) \otimes \mathfrak{D}^u_{\rho\oplus 0}(X)) \to K_*(\mathfrak{D}^u_{\rho\oplus 0}(X)/\mathfrak{C}^u_{\rho\oplus 0}(X)).$$

Using Paschke duality we may define the cap product as the composition

$$\begin{aligned} K_u^p(X) \otimes K_q^u(X;\rho\oplus 0) &= K_{-p}(C_u(X)) \otimes K_{1+q}(\mathfrak{D}^u_{\rho\oplus 0}(X)) \\ &\to K_{-p+1+q}(C_u(X) \otimes \mathfrak{D}^u_{\rho\oplus 0}(X)) \\ &\stackrel{\sigma_*}{\to} K_{-p+1+q}(\mathfrak{D}^u_{\rho\oplus 0}(X)/\mathfrak{C}^u_{\rho\oplus 0}(X)) \\ &\stackrel{(4.4)}{\cong} K_{-p+1+q}(\mathfrak{D}^u_{\rho\oplus 0}(X)) \\ &= K^u_{q-p}(X;\rho\oplus 0), \end{aligned}$$

where the first arrow is the external product on K-theory. So we get the cap product

$$\cap\colon K_u^p(X) \otimes K_q^u(X) \to K^u_{q-p}(X).$$

Let us state in a proposition some properties of it that we will need. The proofs of these properties are analogous to the non-uniform case.

Proposition 4.27. *The cap product has the following properties:*

- *the pairing of $K_u^0(X)$ with $K_*^u(X)$ coincides with the one in Lemma 4.21,*
- *the fact that $K_*^u(X)$ is a module over $K_u^0(X)$ generalizes to*

  $$(P \otimes Q) \cap T = P \cap (Q \cap T) \tag{4.6}$$

  for all elements $P, Q \in K_u^*(X)$ and $T \in K_*^u(X)$, where $\otimes$ is the internal product on uniform K-theory,
- *if X and Y have jointly bounded geometry, then we have the following compatibility with the external products:*

  $$(P \times Q) \cap (S \times T) = (-1)^{qs}(P \cap S) \times (Q \cap T), \tag{4.7}$$

  where $P \in K_u^p(X)$, $Q \in K_u^q(Y)$ and $S \in K_s^u(X)$, $T \in K_t^u(Y)$, and
- *if we have a manifold of bounded geometry M, a vector bundle of bounded geometry $E \to M$ and an operator D of Dirac type, then*

  $$[E] \cap [D] = [D_E] \in K_*^u(M), \tag{4.8}$$

  where D_E is the twisted operator.

4.4 Uniform K-Poincaré duality

We will prove in this section that uniform K-theory is indeed the dual theory to uniform K-homology. To this end we will show that they are Poincaré dual to each other. This will be accomplished by a suitable Mayer–Vietoris induction of which the idea will also be used later in this paper to prove similar results like the uniform Chern character isomorphism theorems in Section 5.3.

Theorem 4.28. *Let M be an m-dimensional spin^c manifold of bounded geometry and without boundary.*

Then the cap product $\cdot \cap [M] \colon K_u^*(M) \to K_{m-*}^u(M)$ with its uniform K-fundamental class $[M] \in K_m^u(M)$ is an isomorphism.

We will need the following Theorem 4.31 about manifolds of bounded geometry. To state it, we have to recall some notions:

Definition 4.29 (Bounded geometry simplicial complexes)**.** A simplicial complex has *bounded geometry* if there is a uniform bound on the number of simplices in the link of each vertex.

A subdivision of a simplicial complex of bounded geometry is called a *uniform subdivision* if

- each simplex is subdivided a uniformly bounded number of times on its n-skeleton, where the n-skeleton is the union of the n-dimensional sub-simplices of the simplex, and
- the distortion $\operatorname{length}(e) + \operatorname{length}(e)^{-1}$ of each edge e of the subdivided complex is uniformly bounded in the metric given by barycentric coordinates of the original complex.

Definition 4.30 (Continuous quasi-isometries)**.** Two metric spaces X and Y are said to be *quasi-isometric* if there is a homeomorphism $f \colon X \to Y$ with

$$\tfrac{1}{C} d_X(x, x') \le d_Y(f(x), f(x')) \le C d_X(x, x')$$

for all $x, x' \in X$ and some constant $C > 0$.

Theorem 4.31 ([Att94, Theorem 1.14])**.** *Let M be a manifold of bounded geometry and without boundary.*

Then M admits a triangulation as a simplicial complex of bounded geometry whose metric given by barycentric coordinates is quasi-isometric to the metric on M induced by the Riemannian structure. This triangulation is unique up to uniform subdivision.

Conversely, if M is a simplicial complex of bounded geometry which is a triangulation of a smooth manifold, then this smooth manifold admits a metric of bounded geometry with respect to which it is quasi-isometric to M.

Remark 4.32. Attie uses in [Att94] a weaker notion of bounded geometry as we do: additionally to a uniformly positive injectivity radius he only requires the sectional curvatures to be bounded in absolute value (i.e., the curvature tensor is bounded in norm), but he assumes nothing about the derivatives (see [Att94, Definition 1.4]). But going into his proof of [Att94, Theorem 1.14], we see that the Riemannian metric constructed for the second statement of the theorem is actually of bounded geometry in our strong sense (i.e., also with bounds on the derivatives of the curvature tensor).

As a corollary we get that for any manifold of bounded geometry in Attie's weak sense there is another Riemannian metric of bounded geometry in our strong sense that is quasi-isometric the original one (in fact, this quasi-isometry is just the identity map of the manifold, as can be seen from the proof).

Lemma 4.33. *Let M be a manifold of bounded geometry.*

Then there is an $\varepsilon > 0$ and a countable collection of uniformly discretely distributed points $\{x_i\} \subset M$ such that $\{B_\varepsilon(x_i)\}$ is a uniformly locally finite cover of M.

Furthermore, it is possible to partition $\mathbb{N}$ into a finite amount of subsets $I_1, \ldots, I_N$ such that for each $1 \le j \le N$ the subset $U_j := \bigcup_{i \in I_j} B_\varepsilon(x_i)$ is a disjoint union of balls that are a uniform distance apart from each other, and such that for each $1 \le K \le N$ the connected components of $U_K := U_1 \cup \ldots \cup U_k$ are also a uniform distance apart from each other (see Figure 4).

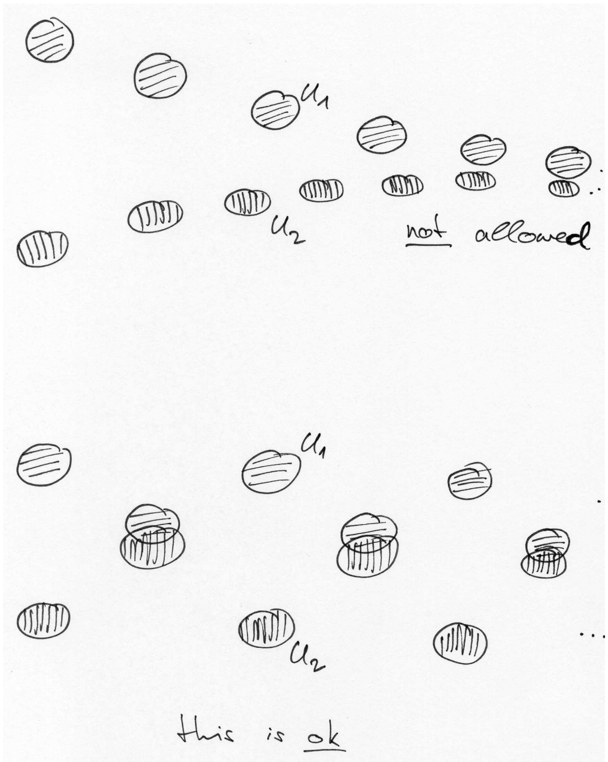

Figure 4: Illustration for Lemma 4.33.

Proof. We triangulate M via the above Theorem 4.31. Then we may take the vertices of this triangulation as our collection of points $\{x_i\}$ and set ε to 2/3 of the length of an edge multiplied with the constant C which we get since the metric derived from barycentric coordinates is quasi-isometric to the metric derived from the Riemannian structure.

Two balls $B_\varepsilon(x_i)$ and $B_\varepsilon(x_j)$ for $x_i \neq x_j$ intersect if and only if x_i and x_j are adjacent vertices, and in the case that they are not adjacent, these balls are a uniform distance apart from each other. Hence it is possible to find a coloring of all these balls $\{B_\varepsilon(x_i)\}$ with finitely many colors having the claimed property.[70] □

Our proof of Poincaré duality is a suitable version of a Mayer–Vietoris induction which will have only finitely many steps. So we first have to discuss the corresponding Mayer–Vietoris sequences.

We will start with the Mayer–Vietoris sequence for uniform K-theory. Let $O \subset M$ be an open subset, not necessarily connected. We denote by (M, d) the metric space M endowed with the metric induced from the Riemannian metric g on M, and by $C_u(O, d)$ we denote the C^*-algebra of all bounded, uniformly continuous functions on O, where we regard O as a metric space equipped with the subset metric induced from d (i.e., we do not equip O with the induced Riemannian metric and consider then the corresponding induced metric structure; see Figure 5).

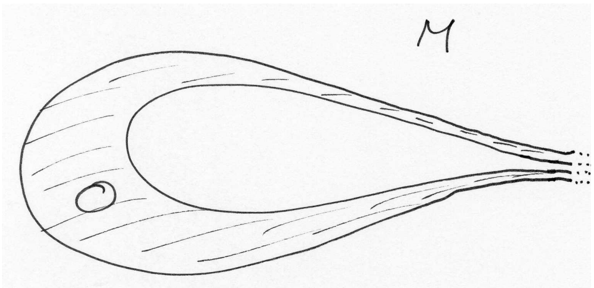

Figure 5: The metric on O induced from being a subset of the metric space (M, d) may differ vastly from the induced Riemannian metric.

Definition 4.34. Let $O \subset M$ be an open subset, not necessarily connected. We define $K_u^p(O \subset M) := K_{-p}(C_u(O, d))$.

We will also need the following technical theorem about the existence of extensions of uniformly continuous functions:

Lemma 4.35. *Let $O \subset M$ be open, not necessarily connected. Then every function $f \in C_u(O, d)$ has an extension to an $F \in C_u(M, d)$.*

Proof. For a metric space X let uX denote the Gelfand space of $C_u(X)$, i.e., this is a compactification of X (the *Samuel compactification*) with the following universal property: a bounded, continuous function f on X has an extension to a continuous function on uX

[70] see Footnote 65

if and only if f is uniformly continuous. We will use the following property of Samuel compactifications (see [Woo95, Theorem 2.9]): if $S \subset X \subset uX$, then the closure $\mathrm{cl}_{uX}(S)$ of S in uX is the Samuel compactification uS of S.

So given $f \in C_u(O, d)$, we can extend it to a continuous function $\tilde{f} \in C(uO)$. Since $uO = \mathrm{cl}_{uM}(O)$, i.e., a closed subset of a compact Hausdorff space, we can extend $\tilde{f}$ by the Tietze extension theorem to a bounded, continuous function $\tilde{F}$ on uM. Its restriction $F := \tilde{F}|_M$ to M is then a bounded, uniformly continuous function of M extending f. □

Lemma 4.36. *Let the subsets U_j, U_K of M for $1 \le j, K \le N$ be as in Lemma 4.33. Then we have Mayer–Vietoris sequences*

$$\begin{tikzcd} K_u^0(U_K \cup U_{k+1} \subset M) \arrow[r] & K_u^0(U_K \subset M) \oplus K_u^0(U_{k+1} \subset M) \arrow[r] & K_u^0(U_K \cap U_{k+1} \subset M) \arrow[d] \\ K_u^1(U_K \cap U_{k+1} \subset M) \arrow[u] & K_u^1(U_K \subset M) \oplus K_u^1(U_{k+1} \subset M) \arrow[l] & K_u^1(U_K \cup U_{k+1} \subset M) \arrow[l] \end{tikzcd}$$

where the horizontal arrows are induced from the corresponding restriction maps.

Proof. Recall the Mayer–Vietoris sequence for operator K-theory of C^*-algebras (see, e.g., [Bla98, Theorem 21.2.2]): given a commutative diagram of C^*-algebras

$$\begin{tikzcd} P \arrow[r, "\sigma_1"] \arrow[d, "\sigma_2"'] & A_1 \arrow[d, "\varphi_1"] \\ A_2 \arrow[r, "\varphi_2"] & B \end{tikzcd}$$

with $P = \{(a_1, a_2) \mid \varphi_1(a_1) = \varphi_2(a_2)\} \subset A_1 \oplus A_2$ and φ_1 and φ_2 surjective, then there is a long exact sequence (via Bott periodicity we get the 6-term exact sequence)

$$\ldots \to K_n(P) \xrightarrow{(\sigma_{1*}, \sigma_{2*})} K_n(A_1) \oplus K_n(A_2) \xrightarrow{\varphi_{2*} - \varphi_{1*}} K_n(B) \to K_{n-1}(P) \to \ldots$$

We set $A_1 := C_u(U_K, d)$, $A_2 := C_u(U_{k+1}, d)$, $B := C_u(U_K \cap U_{k+1}, d)$ and φ_1, φ_2 the corresponding restriction maps. Due to the property of the sets U_K as stated in the Lemma 4.33 we get $P = C_u(U_K \cup U_{k+1}, d)$ and σ_1, σ_2 again just the restriction maps. To show that the maps φ_1 and φ_2 are surjective we have to use the above Lemma 4.35. □

We will also need corresponding Mayer–Vietoris sequences for uniform K-homology. As for uniform K-theory we use here also the induced subspace metric (and not the metric derived from the induced Riemannian metric): let a not necessarily connected subset $O \subset M$ be given. We define $K_*^u(O \subset M)$ to be the uniform K-homology of O, where O is equipped with the subspace metric from M, where we view M as a metric space. The inclusion $O \hookrightarrow M$ is in general not a proper map (e.g., if O is an open ball in a manifold) but this is no problem to us since we will have to use the wrong-way maps that exist for open subsets $O \subset M$: they are given by the inclusions $L\text{-}\operatorname{Lip}_R(O) \subset L\text{-}\operatorname{Lip}_R(M)$ for all $R, L > 0$. So we get a map $K_*^u(M) \to K_*^u(O \subset M)$ for every open subset $O \subset M$.

Existence of Mayer–Vietoris sequences for uniform K-homology of the subsets in the cover $\{U_K, U_{k+1}\}$ of $U_K \cup U_{k+1}$ (recall that we used Lemma 4.33 to get these subsets) incorporating the wrong-way maps may be similarly shown as [HR00, Section 8.5]. The crucial excision isomorphism from that section may be constructed analogously as described in [HR00, Footnote 73]: for that construction Kasparov's Technical Theorem is used, and we have to use here in our uniform case the corresponding uniform construction which is as used in our construction of the external product for uniform K-homology.

Note that Špakula constructed a Mayer–Vietoris sequence for uniform K-homology in [Špa09, Section 5], but for closed subsets of a proper metric space. His arrows also go in the other direction as ours (since his arrows are induced by the usual functoriality of uniform K-homology).

We denote by $[M]|_O \in K_m^u(O \subset M)$ the class of the Dirac operator associated to the restriction to a neighbourhood of O of the complex spinor bundle of bounded geometry defining the spin^c-structure of M (i.e., we equip the neighbourhood with the induced spin^c-structure).

The cap product of $K_u^*(O \subset M)$ with $[M]|_O$ is analogously defined as the usual one, i.e., we get maps $\cdot \cap [M]|_O \colon K_u^*(O \subset M) \to K_{m-*}^u(O \subset M)$. Now we have to argue why we get commutative squares between the Mayer–Vietoris sequences of uniform K-theory and uniform K-homology using the cap product. This is known for usual K-theory and K-homology; see, e.g., [HR00, Exercise 11.8.11(c)]. Since the cap product is in our uniform case completely analogously defined (see the second-to-last display before Proposition 4.27), we may analogously conclude that we get commutative squares between our uniform Mayer–Vietoris sequences.

Let us summarize the above results:

Lemma 4.37. *Let the subsets U_j, U_K of M for $1 \le j, K \le N$ be as in Lemma 4.33. Then we have corresponding Mayer–Vietoris sequences*

$$\begin{tikzcd}
K_0^u(U_K \cup U_{k+1} \subset M) \arrow[r] & K_0^u(U_K \subset M) \oplus K_0^u(U_{k+1} \subset M) \arrow[r] & K_0^u(U_K \cap U_{k+1} \subset M) \arrow[d] \\
K_1^u(U_K \cap U_{k+1} \subset M) \arrow[u] & K_1^u(U_K \subset M) \oplus K_1^u(U_{k+1} \subset M) \arrow[l] & K_1^u(U_K \cup U_{k+1} \subset M) \arrow[l]
\end{tikzcd}$$

and the cap product gives the following commutative diagram:

$$\begin{tikzcd}
K_u^*(U_K \cap U_{k+1} \subset M) \arrow[r] \arrow[d] & K_u^*(U_K \cup U_{k+1} \subset M) \arrow[r] \arrow[d] & K_u^*(U_K \subset M) \oplus K_u^*(U_{k+1} \subset M) \arrow[d] \arrow[ll, bend right] \\
K_{m-*}^u(U_K \cap U_{k+1} \subset M) \arrow[r] & K_{m-*}^u(U_K \cup U_{k+1} \subset M) \arrow[r] & K_{m-*}^u(U_K \subset M) \oplus K_{m-*}^u(U_{k+1} \subset M) \arrow[ll, bend left]
\end{tikzcd}$$

(We have suppressed the index shift due to the boundary maps in the latter diagram.)

Let us begin the proof of Poincaré duality. First we invoke Lemma 4.33 to get subsets U_j for $1 \le j \le N$.

The induction starts with the subsets U_1, U_2 and $U_1 \cap U_2$, which are collections of uniformly discretely distributed open balls, resp., in the case of $U_1 \cap U_2$ it is a collection of intersections of open balls, which is homotopy equivalent to a collection of uniformly discretely distributed open balls by a uniformly cobounded, proper and Lipschitz homotopy. Now uniform K-theory of a space coincides with the uniform K-theory of its completion, and furthermore, uniform K-theory is homotopy invariant with respect to Lipschitz homotopies. So the uniform K-theory of a collection of open balls is the same as the uniform K-theory of a collection of points. This groups we have already computed in Lemma 4.4.

Uniform K-homology is homotopy invariant with respect to uniformly cobounded, proper and Lipschitz homotopies (see Theorem 3.26), and for totally bounded spaces it coincides with usual K-homology (see Proposition 3.7). So we have to compute uniform K-homology of a collection of uniformly discretely distributed open balls. Note that we have already computed the uniform K-homology of uniformly discretely distributed points in Lemma 3.17.

Lemma 4.38. *Let M be an m-dimensional manifold of bounded geometry and let $U \subset M$ be a subset consisting of uniformly discretely distributed geodesic balls in M having radius less than the injectivity radius of M (i.e., each geodesic ball is diffeomorphic to the standard ball in Euclidean space $\mathbb{R}^m$). Let the balls be indexed by a set Y.*

Then we have $K_m^u(U \subset M) \cong \ell^\infty_{\mathbb{Z}}(Y)$, the group of all bounded, integer-valued sequences indexed by Y, and $K_p^u(U \subset M) = 0$ for $p \neq m$. An analogous statement holds for uniform K-homology.

Proof. The proof is analogous to the proof of Lemma 3.17. It uses the fact that for an open Ball $O \subset \mathbb{R}^m$ we have $K_m(O \subset \mathbb{R}^m) \cong \mathbb{Z}$, and $K_p(O \subset \mathbb{R}^m) = 0$ for $p \neq m$. □

Now we can argue that cap product is an isomorphism $K_u^*(U \subset M) \cong K_{m-*}^u(U \subset M)$, where U is as in the above lemma. For this we have to note that if M is a spin^c manifold, then the restriction of its complex spinor bundle to any ball of U is isomorphic to the complex spinor bundle on the open ball $O \subset \mathbb{R}^m$. This means that the cap product on U coincides on each open ball of U with the usual cap product on the open ball $O \subset \mathbb{R}^m$. This all shows that we have Poincaré duality for the subsets U_1, U_2 and $U_1 \cap U_2$ (note that $U_1 \cap U_2$ is homotopic to a collection of open balls).

With the above Lemma 4.37 we therefore get with the five lemma that the cap product is also an isomorphism for $U_1 \cup U_2$. The rest of the proof proceeds by induction over k (there are only finitely many steps since we only go up to $k = N-1$), invoking every time the above Lemma 4.37 and the five lemma. Note that in order to see that the cap product is an isomorphism on $U_K \cap U_{k+1}$, we have to write $U_K \cap U_{k+1} = (U_1 \cap U_{k+1}) \cup \ldots \cup (U_k \cap U_{k+1})$. This is a union of k geodesically convex open sets and we have to do a separate induction on this one.

5 Index theorems for uniform operators

In this section we will assemble everything that we had up to now into our uniform index theorems. For this we will first have to define the uniform de Rham (co-)homology theories that will serve as receptacles for the index classes of our operators (the uniform homological Chern character of a uniform, abstractly elliptic operator will give us its analytic index class and uniform de Rham cohomology will receive the topological index classes of elliptic uniform pseudodifferential operators). After a small detour into the world of the Chern character isomorphism theorem we will then finally prove the uniform index theorems.

5.1 Cyclic cocycles of uniformly finitely summable modules

The goal of this section is to construct the uniform homological Chern character maps from uniform K-homology $K_*^u(M)$ of M to continuous periodic cyclic cohomology $HP^*_{\mathrm{cont}}(W^{\infty,1}(M))$ of the Sobolev space $W^{\infty,1}(M)$.

First we will recall the definition of Hochschild, cyclic and periodic cyclic cohomology of a (possibly non-unital) complete locally convex algebra A[71]. The classical reference for this is, of course, Connes' seminal paper [Con85]. The author also found Khalkhali's book [Kha13] a useful introduction to these matters.

Definition 5.1. The *continuous Hochschild cohomology* $HH^*_{\mathrm{cont}}(A)$ of A is the homology of the complex

$$C^0_{\mathrm{cont}}(A) \xrightarrow{b} C^1_{\mathrm{cont}}(A) \xrightarrow{b} \ldots,$$

where $C^n_{\mathrm{cont}}(A) = \mathrm{Hom}(A^{\widehat{\otimes}(n+1)}, \mathbb{C})$ and the boundary map b is given by

$$\begin{aligned}(b\varphi)(a_0, \ldots, a_{n+1}) = &\sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + \\ &+ (-1)^{n+1} \varphi(a_{n+1}a_0, a_1, \ldots, a_n).\end{aligned}$$

We use the completed projective tensor product $\widehat{\otimes}$ and the linear functionals are assumed to be continuous. But we still factor our only the image of the boundary operator to define the homology, and *not* the closure of the image of b.

Definition 5.2. The *continuous cyclic cohomology* $HC^*_{\mathrm{cont}}(A)$ of A is the homology of the following subcomplex of the Hochschild cochain complex:

$$C^0_{\lambda,\mathrm{cont}}(A) \xrightarrow{b} C^1_{\lambda,\mathrm{cont}}(A) \xrightarrow{b} \ldots,$$

where $C^n_{\lambda,\mathrm{cont}}(A) = \{\varphi \in C^n_{\mathrm{cont}}(A) \colon \varphi(a_n, a_0, \ldots, a_{n-1}) = (-1)^n \varphi(a_0, a_1, \ldots, a_n)\}$.

[71]We consider here only algebras over the field $\mathbb{C}$. Furthermore, we assume that multiplication in A is jointly continuous.

There is a *periodicity operator* $S\colon HC^n_{\mathrm{cont}}(A) \to HC^{n+2}_{\mathrm{cont}}(A)$. For the tedious definition of it on the level of cyclic cochains we refer the reader to [Con85, Lemma 11 on p. 322]. Note that we have to use slightly different normalizations constants for S due to the fact that we want S still to be compatible with the Chern–Connes character (to be defined in a moment) for which we use slightly different normalizations constants as Connes does.

Definition 5.3. The *continuous periodic cyclic cohomology* $HP^*_{\mathrm{cont}}(A)$ of A is defined as the direct limit

$$HP^*_{\mathrm{cont}}(A) = \varinjlim HC^{*+2n}_{\mathrm{cont}}(A)$$

with respect to the maps S.

Let (H, ρ, T) be a graded uniform Fredholm module over M and denote by ϵ the grading automorphism of H. Furthermore, assume that (H, ρ, T) is involutive and uniformly p-summable, i.e., $\sup_{f \in L\text{-}\mathrm{Lip}_R(M)} \|[T, \rho(f)]\|_p < \infty$ for the Schatten p-norm $\|\cdot\|_p$.

Having such a module at hand we define for all m with $2m+1 \geq p$ a cyclic $2m$-cocycle on $W^{\infty,1}(M)$, i.e., on the Sobolev space of infinite order and L^1-integrability, by

$$\mathrm{ch}^{0,2m}(H, \rho, T)(f_0, \ldots, f_{2m}) := (-1)^m (2\pi i)^m \frac{1}{2} \frac{m!}{(2m)!} \operatorname{tr}\big(\epsilon T [T, f_0] \cdots [T, f_{2m}]\big).$$

We have the compatibility $S \circ \mathrm{ch}^{0,2m} = \mathrm{ch}^{0,2m+2}$ and therefore we get a map

$$\mathrm{ch}^0\colon K_0^u(M) \dashrightarrow HP^0_{\mathrm{cont}}(W^{\infty,1}(M)).$$

The dashed arrow indicates that we do not know that every uniform, even K-homology class is represented by a uniformly finitely summable module, and we also do not know if the map is well-defined, i.e., if two such modules representing the same K-homology class will be mapped to the same cyclic cocycle class. For spin^c manifolds the first mentioned problem is solved by Poincaré duality which states that every uniform K-homology class may be represented by the difference of two twisted Dirac operators (which are uniformly finitely summable). But the second mentioned problem about the well-definedness is much more serious and will only be solved by the local index theorem. We will state the resolution of this problem in Corollary 5.21.

Given an ungraded, involutive, uniformly p-summable Fredholm module (H, ρ, T), we define for all m with $2m \geq p$ a cyclic $(2m-1)$-cocycle on $W^{\infty,1}(M)$ by

$$\begin{aligned}\mathrm{ch}^{1,2m-1}&(H, \rho, T)(f_0, \ldots, f_{2m-1}) = \\ &= (-1)^m (2\pi i)^m \tfrac{1}{2} (2m-1)(2m-3) \cdots 3 \cdot 1 \operatorname{tr}\big(T [T, f_0] \cdots [T, f_{2m-1}]\big).\end{aligned}$$

Again, this definition is compatible with the periodicity operator S and so defines a map

$$\mathrm{ch}^1\colon K_1^u(M) \dashrightarrow HP^1_{\mathrm{cont}}(W^{\infty,1}(M)).$$

5.2 Uniform de Rham (co-)homology

We will start with defining and discussing uniform de Rham homology and afterwards we will get to the bounded, resp. uniform, de Rham cohomology.

Definition 5.4. The space of *uniform de Rham p-currents* $\Omega_p^u(M)$ is defined as the dual space of the Fréchet space $W^{\infty,1}(\Omega^p(M))$, i.e.,

$$\Omega_p^u(M) := \operatorname{Hom}(W^{\infty,1}(\Omega^p(M)), \mathbb{C}).$$

Since the exterior derivative $d\colon W^{\infty,1}(\Omega^p(M)) \to W^{\infty,1}(\Omega^{p+1}(M))$ is continuous we get a corresponding dual differential

$$d\colon \Omega_p^u(M) \to \Omega_{p-1}^u(M).$$

The *uniform de Rham homology* $H_*^{u,\mathrm{dR}}(M)$ with coefficients in $\mathbb{C}$ is defined as the homology of the complex

$$\ldots \xrightarrow{d} \Omega_p^u(M) \xrightarrow{d} \Omega_{p-1}^u(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega_0(M) \to 0.$$

Definition 5.5. We define a map $\alpha\colon C_{\mathrm{cont}}^p(W^{\infty,1}(M)) \to \Omega_p^u(M)$ by

$$\alpha(\varphi)(f_0 df_1 \wedge \ldots \wedge df_p) := \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (-1)^\sigma \varphi(f_0, f_{\sigma(1)}, \ldots, f_{\sigma(p)}).$$

The antisymmetrization that we have done in the above definition of α maps Hochschild cocycles to Hochschild cocycles and vanishes on Hochschild coboundaries. This means that α descends to a map

$$\alpha\colon HH_{\mathrm{cont}}^*(W^{\infty,1}(M)) \to \Omega_*^u(M)$$

on Hochschild cohomology.

Theorem 5.6. *For any Riemannian manifold M of bounded geometry and without boundary the map $\alpha\colon HH_{\mathrm{cont}}^p(W^{\infty,1}(M)) \to \Omega_p^u(M)$ is an isomorphism for all p.*

Proof. The proof is analogous to the one given in [Con85, Lemma 45a on page 128] for the case of compact manifolds. We describe here only the places where we have to adjust it for non-compact manifolds.

The proof in [Con85] relies heavily on Lemma 44 there. First note that direct sums, tensor products and duals of vector bundles of bounded geometry are again of bounded geometry. Since the tangent and cotangent bundle of a manifold of bounded geometry have, of course, bounded geometry, the bundles E_k occuring in Lemma 44 of [Con85] have bounded geometry.

Since M is non-compact, there always exists a nowhere vanishing vector field on it. For the proof of [Con85, Lemma 44] to work we need that the vector field X constructed there is in $C_b^\infty(E_1^*)$, i.e., it and all its derivatives must be bounded in sup-norm. This can be achieved since M has bounded geometry.

Furthermore, we need the statement $W^{\infty,1}(M) \mathbin{\hat\otimes} W^{\infty,1}(M) \cong W^{\infty,1}(M \times M)$. For this we refer to the answer [Mic14] of P. Michor on MathOverflow.

The fact that the modules $\mathcal{M}_k = W^{\infty,1}(M \times M, E_k)$ are topologically projective, i.e., are direct summands of topological modules of the form $\mathcal{M}'_k = W^{\infty,1}(M \times M) \mathbin{\hat\otimes} \mathcal{E}_k$, where $\mathcal{E}_k$ are complete locally convex vector spaces, follows from the fact that every vector bundle F of bounded geometry is C_b^∞-complemented, i.e., there is a vector bundle G of bounded geometry such that $F \oplus G$ is C_b^∞-isomorphic to a trivial bundle with the flat connection. This is our Proposition 4.12.

With the above notes in mind, the proof of [Con85, Lemma 45a on page 128] for the case of compact manifolds works also for non-compact manifolds in our setting here. If there are constructions to be done in the proof we have to do them uniformly (e.g., controlling derivatives uniformly in the points of the manifold) by using the bounded geometry of M. □

The inverse map $\beta\colon \Omega_p^u(M) \to HH^p_{\mathrm{cont}}(W^{\infty,1}(M))$ of α is given by

$$\beta(C)(f_0, f_1, \ldots, f_p) = C(f_0 df_1 \wedge \ldots \wedge df_p).$$

Now the proofs of Lemma 45b and Theorem 46 in [Con85] translate without change to our setting here so that we finally get:

Theorem 5.7. *Let M be a Riemannian manifold of bounded geometry and without boundary.*

For each $n \in \mathbb{N}_0$ the continuous cyclic cohomology $HC^n_{\mathrm{cont}}(W^{\infty,1}(M))$ is canonically isomorphic to

$$Z_n^u(M) \oplus H^{u,\mathrm{dR}}_{n-2}(M) \oplus H^{u,\mathrm{dR}}_{n-4}(M) \oplus \ldots,$$

where $Z_n^u(M) \subset \Omega_n^u(M)$ is the subspace of closed currents.

The periodicity operator $S\colon HC^n_{\mathrm{cont}}(W^{\infty,1}(M)) \to HC^{n+2}_{\mathrm{cont}}(W^{\infty,1}(M))$ is given under the above isomorphism as the map that sends cycles of $Z_n^u(M)$ to their homology classes.

And last, since periodic cyclic cohomology is the direct limit of cyclic cohomology, we finally get

$$\alpha_*\colon HP^{\mathrm{ev/odd}}_{\mathrm{cont}}(W^{\infty,1}(M)) \xrightarrow{\cong} H^{u,\mathrm{dR}}_{\mathrm{ev/odd}}(M).$$

We denote this isomorphism by α_* since it is induced from the map α defined above.

Let us now get to the dual cohomology theory to uniform de Rham homology.

Definition 5.8 (Bounded de Rham cohomology)**.** Let $\Omega_b^p(M)$ denote the vector space of p-forms on M, which are bounded in the norm

$$\|\gamma\| := \sup_{x \in M}\{\|\gamma(x)\| + \|d\gamma(x)\|\}.$$

The *bounded de Rham cohomology* $H^*_{b,\mathrm{dR}}(M)$ is defined as the homology of the corresponding complex.

The Poincaré duality map between bounded de Rham cohomology and uniform de Rham homology is defined as the map induced by the following one on forms:

$$\Omega_b^p(M) \to \Omega_{m-p}^u(M),\ \gamma \mapsto \big(\omega \mapsto \int_M \omega \wedge \gamma\big).$$

Theorem 5.9 (De Rham Poincaré duality)**.** *Let M^m be an oriented Riemannian manifold of bounded geometry and without boundary.*

Then the Poincaré duality map induces an isomorphism

$$H_{b,\mathrm{dR}}^*(M) \xrightarrow{\cong} H_{m-*}^{u,\mathrm{dR}}(M)$$

between bounded de Rham cohomology of M and uniform de Rham homology of M.

Proof. We use the same idea as for the proof of Poincaré duality in Section 4.4. The details that we do have a suitable Mayer–Vietoris sequence for bounded de Rham cohomology may be found in the author's Ph.D. thesis [Eng14, Section 5.6] and the arguments for uniform de Rham homology are similar. That the above defined Poincaré duality map is a natural transformation from one Mayer–Vietoris sequence to the other may be proved analogously as in the case of compact manifolds; see, e.g., [Lee03, Exercise 16-6]. □

For proving that the Chern character (which we will discuss later in this section) induces an isomorphism modulo torsion, we will have to use a different model for bounded de Rham cohomology, namely uniform de Rham cohomology.

Definition 5.10. The *uniform de Rham cohomology* $H_{u,\mathrm{dR}}^*(M)$ of a Riemannian manifold M of bounded geometry is defined by using the complex of uniform C^∞-spaces[72] $C_b^\infty(\Omega^\bullet(M))$, i.e., differential forms on M which have in normal coordinates bounded coefficient functions and all derivatives of them are also bounded.

Proposition 5.11. *Let M be a manifold of bounded geometry and without boundary. Then we have*

$$H_{u,\mathrm{dR}}^*(M) \cong H_{b,\mathrm{dR}}^*(M).$$

Proof. We use a Mayer–Vietoris argument as in the proof of the uniform K-Poincaré duality in Section 4.4. The only thing to show here is that we do have such Mayer–Vietoris sequences for both the bounded and the uniform de Rham cohomology.

We use for $H_{b,\mathrm{dR}}^*(M)$ the spaces $\Omega_b^\bullet(O \subset M)$ of bounded forms on the open subset $O \subset M$ which have an extension to all of M. For this groups we have exactness of the short sequence

$$0 \to \Omega_b^p(U_K \cup U_{k+1} \subset M) \to \Omega_b^p(U_K \subset M) \oplus \Omega_b^p(U_{k+1} \subset M) \to \Omega_b^p(U_K \cap U_{k+1} \subset M) \to 0$$

for the open subsets U_j and U_K of M from Lemma 4.33 (the non-trivial part in showing the exactness is the surjectivity of the last non-trivial map) and therefore we get corresponding Mayer–Vietoris sequences.

For uniform de Rham cohomology it is even easier since here we automatically may extend uniform differential forms to all of M due to the boundedness of the coefficient functions and all their derivatives. □

[72]see Definition 2.7

Theorem 5.12 (Existence of the uniform Chern character)**.** *Let M be a Riemannian manifold of bounded geometry and without boundary.*

Then we have a ring homomorphism $\operatorname{ch}\colon K_u^*(M) \to H_{u,\mathrm{dR}}^*(M)$ *with*

$$\operatorname{ch}(K_u^0(M)) \subset H_{u,\mathrm{dR}}^{\mathrm{ev}}(M) \text{ and } \operatorname{ch}(K_u^1(M)) \subset H_{u,\mathrm{dR}}^{\mathrm{odd}}(M).$$

Proof. The Chern character is defined via Chern–Weil theory. That we get uniform forms if we use vector bundles of bounded geometry is proved in [Roe88a, Theorem 3.8] and so we get a map $\operatorname{ch}\colon K_u^0(M) \to H_{u,\mathrm{dR}}^{\mathrm{ev}}(M)$. That we also have a map $\operatorname{ch}\colon K_u^1(M) \to H_{u,\mathrm{dR}}^{\mathrm{odd}}(M)$ uses the description of $K_u^1(M)$ as consisting of vector bundles over $S^1 \times M$ and a corresponding suspension isomorphism for uniform de Rham cohomology. Details (for bounded cohomology, but for uniform cohomology it is analogous) may be found in the author's Ph.D. thesis [Eng14, Sections 5.4, 5.5]. □

5.3 Uniform Chern character isomorphism theorems

We have constructed Chern characters $K_u^*(M) \to H_{u,\mathrm{dR}}^*(M)$ and $K_*^u(M) \to H_*^{u,\mathrm{dR}}(M)$, where we already use Corollary 5.21 which states that the uniform homological Chern character is well-defined. In the compact case the Chern characters are isomorphisms modulo torsion and it is natural to ask the same question here in the uniform setting. It is the goal of this section to answer this question positively.

The proofs use the same Mayer–Vietoris induction as the proof of Poincaré duality in Section 4.4. Therefore we will discuss in this section only the parts of the proofs which need additional arguments.

The most crucial detail to discuss is the statement of the theorem itself since we cannot just take the tensor product of the K-groups with the complex numbers to get isomorphisms, but we additionally have to form a certain completion. We will discuss this completion directly after the statement of the theorem.

Theorem 5.13. *Let M be a manifold of bounded geometry and without boundary. Then the Chern characters induce linear, continuous isomorphisms*[73]

$$K_u^*(M) \mathbin{\bar{\otimes}} \mathbb{C} \cong H_{u,\mathrm{dR}}^*(M) \text{ and } K_*^u(M) \mathbin{\bar{\otimes}} \mathbb{C} \cong H_*^{u,\mathrm{dR}}(M).$$

Let us discuss why we have to take a completion at all. Consider the beginning of the Mayer–Vietoris induction where we have to show that the Chern characters induce isomorphisms on a countably infinite collection of uniformly discretely distributed points. Let these points be indexed by a set Y. Then the K-groups of Y are given by $\ell_{\mathbb{Z}}^\infty(Y)$, the group of all bounded, integer-valued sequences indexed by Y, and the de Rham groups are given by $\ell^\infty(Y)$, the group of all bounded, complex valued sequences on Y. But since Y is countably infinite we have $\ell_{\mathbb{Z}}^\infty(Y) \otimes \mathbb{C} \ncong \ell^\infty(Y)$. Instead we have $\overline{\ell_{\mathbb{Z}}^\infty(Y) \otimes \mathbb{C}} \cong \ell^\infty(Y)$.

[73]The inverse maps are in general not continuous since $H_{u,\mathrm{dR}}^*(M)$, resp. $H_*^{u,\mathrm{dR}}(M)$, are in general (e.g., if M is not compact) not Hausdorff, whereas $K_u^*(M) \mathbin{\bar{\otimes}} \mathbb{C}$, resp. $K_*^u(M) \mathbin{\bar{\otimes}} \mathbb{C}$, are. The topology on the latter spaces is defined by equipping the K-groups with the discrete topology and then forming the completed tensor product with $\mathbb{C}$ which will be discussed after the statement of the theorem.

To define the *completed topological tensor product of an abelian group with* $\mathbb{C}$ we will need the notion of the *free (abelian) topological group*: if X is any completely regular[74] topological space, then the free topological group $F(X)$ on X is a topological group such that we have

- a topological embedding $X \hookrightarrow F(X)$ of X as a closed subset, so that X generates $F(X)$ algebraically as a free group (i.e., the algebraic group underlying the free topological group on X is the free group on X), and we have
- the following universal property: for every continuous map $\phi\colon X \to G$, where G is any topological group, we have a unique extension $\Phi\colon F(X) \to G$ of ϕ to a continuous group homomorphism on $F(X)$:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\phi"'] & F(X) \arrow[dl, dashed, "\exists!\Phi"] \\ G & \end{tikzcd}$$

The free abelian topological group $A(X)$ has the corresponding analogous properties. Furthermore, the commutator subgroup $[F(X), F(X)]$ of $F(X)$ is closed and the quotient $F(X)/[F(X), F(X)]$ is both algebraically and topologically $A(X)$.

As an easy example consider X equipped with the discrete topology. Then $F(X)$ and $A(X)$ also have the discrete topology.

It seems that free (abelian) topological groups were apparently introduced by Markov in [Mar41]. But unfortunately, the author could not obtain any (neither russian nor english) copy of this article. A complete proof of the existence of such groups was given by Markov in [Mar45]. Since his proof was long and complicated, several other authors gave other proofs, e.g., Nakayama in [Nak43], Kakutani in [Kak44] and Graev in [Gra48].

Now let us construct for any abelian topological group G the complete topological vector space $G \bar{\otimes} \mathbb{C}$. We form the topological tensor product $G \otimes \mathbb{C}$ of abelian topological groups in the usual way: we start with the free abelian topological group $A(G \times \mathbb{C})$ over the topological space $G \times \mathbb{C}$ equipped with the product topology[75] and then take the quotient $A(G \times \mathbb{C})/\mathcal{N}$ of it,[76] where $\mathcal{N}$ is the closure of the normal subgroup generated by the usual relations for the tensor product.[77] Now we may put on $G \otimes \mathbb{C}$ the structure of a topological vector space by defining the scalar multiplication to be $\lambda(g \otimes r) := g \otimes \lambda r$.

What we now got is a topological vector space $G \otimes \mathbb{C}$ together with a continuous map $G \times \mathbb{C} \to G \otimes \mathbb{C}$ with the following universal property: for every continuous map

[74]That is to say, every closed set K can be separated with a continuous function from every point $x \notin K$. Note that this does not necessarily imply that X is Hausdorff.

[75]Note that every topological group is automatically completely regular and therefore the product $G \times \mathbb{C}$ is also completely regular.

[76]Since $A(X)$ is both algebraically and topologically the quotient of $F(X)$ by its commutator subgroup, we could also have started with $F(G \times \mathbb{C})$ and additionally put the commutator relations into $\mathcal{N}$.

[77]That is to say, $\mathcal{N}$ contains $(g_1 + g_2) \times r - g_1 \times r - g_2 \times r$, $g \times (r_1 + r_2) - g \times r_1 - g \times r_2$ and $zg \times r - z(g \times r)$, $g \times zr - z(g \times r)$, where $g, g_1, g_2 \in G$, $r, r_1, r_2 \in \mathbb{C}$ and $z \in \mathbb{Z}$.

$\phi\colon G \times \mathbb{C} \to V$ into any topological vector space V and such that ϕ is bilinear[78], there exists a unique, continuous linear map $\Phi\colon G \otimes \mathbb{C} \to V$ such that the following diagram commutes:

$$\begin{tikzcd} G \times \mathbb{C} \arrow[r] \arrow[d, "\phi"'] & G \otimes \mathbb{C} \arrow[dl, dashed, "\exists!\Phi"] \\ V & \end{tikzcd}$$

Since every topological vector space may be completed we do this with $G \otimes \mathbb{C}$ to finally arrive at $G \bar{\otimes} \mathbb{C}$. Since every continuous linear map of topological vector spaces is automatically uniformly continuous, i.e., may be extended to the completion of the topological vector space, $G \bar{\otimes} \mathbb{C}$ enjoys the following universal property which we will raise to a definition:

Definition 5.14 (Completed topological tensor product with $\mathbb{C}$)**.** Let G be an abelian topological group. Then $G \bar{\otimes} \mathbb{C}$ is a complete topological vector space over $\mathbb{C}$ together with a continuous map $G \times \mathbb{C} \to G \bar{\otimes} \mathbb{C}$ that enjoy the following universal property: for every continuous map $\phi\colon G \times \mathbb{C} \to V$ into any complete topological vector space V and such that ϕ is bilinear[79], there exists a unique, continuous linear map $\Phi\colon G \bar{\otimes} \mathbb{C} \to V$ such that the following diagram commutes:

$$\begin{tikzcd} G \times \mathbb{C} \arrow[r] \arrow[d, "\phi"'] & G \bar{\otimes} \mathbb{C} \arrow[dl, dashed, "\exists!\Phi"] \\ V & \end{tikzcd}$$

We will give now two examples for the computation of $G \bar{\otimes} \mathbb{C}$. The first one is easy and just a warm-up for the second which we already mentioned. Both examples are proved by checking the universal property.

Examples 5.15. The first one is $\mathbb{Z} \bar{\otimes} \mathbb{C} \cong \mathbb{C}$.

For the second example consider the group $\ell^\infty_\mathbb{Z}$ consisting of bounded, integer-valued sequences. Then $\ell^\infty_\mathbb{Z} \bar{\otimes} \mathbb{C} \cong \ell^\infty$.

Since we want to use the completed topological tensor product with $\mathbb{C}$ in a Mayer–Vietoris argument, we have to show that it transforms exact sequences to exact sequences.

So we have to show that the functor $G \mapsto G \bar{\otimes} \mathbb{C}$ is exact. But we have to be careful here: though taking the tensor product with $\mathbb{C}$ is exact, passing to completions is usually not—at least if the exact sequence we started with was only algebraically exact. Let us explain this a bit more thoroughly: if we have a sequence of topological vector spaces

$$\ldots \longrightarrow V_i \xrightarrow{\varphi_i} V_{i+1} \xrightarrow{\varphi_{i+1}} V_{i+2} \longrightarrow \ldots$$

which is exact in the algebraic sense (i.e., $\operatorname{im} \varphi_i = \ker \varphi_{i+1}$), and if the maps φ_i are continuous such that they extend to maps on the completions $\overline{V_i}$, we do not necessarily

[78] That is to say, $\phi(\cdot, r)$ is a group homomorphism for all $r \in \mathbb{C}$ and $\phi(g, \cdot)$ is a linear map for all $g \in G$. Note that we then also have $\phi(zg, r) = z\phi(g, r) = \phi(g, zr)$ for all $z \in \mathbb{Z}$, $g \in G$ and $r \in \mathbb{C}$.

[79] see Footnote 78

get that

$$\dots \longrightarrow \overline{V_i} \xrightarrow{\overline{\varphi_i}} \overline{V_{i+1}} \xrightarrow{\overline{\varphi_{i+1}}} \overline{V_{i+2}} \longrightarrow \dots$$

is again algebraically exact. The problem is that though we always have $\overline{\ker \varphi_i} = \ker \overline{\varphi_i}$, we generally only get $\overline{\operatorname{im} \varphi_i} \supset \operatorname{im} \overline{\varphi_i}$. To correct this problem we have to start with an exact sequence which is also topologically exact, i.e., we need that not only $\operatorname{im} \varphi_i = \ker \varphi_{i+1}$, but we also need that φ_i induces a topological isomorphism $V_i/\ker \varphi_i \cong \operatorname{im} \varphi_i$.

To prove that in this case we get $\overline{\operatorname{im} \varphi_i} = \operatorname{im} \overline{\varphi_i}$ we consider the inverse map

$$\psi_i := \varphi_i^{-1} \colon \operatorname{im} \varphi_i \to V_i/\ker \varphi_i.$$

Since ψ_i is continuous (this is the point which breaks down without the additional assumption that φ_i induces a topological isomorphism $V_i/\ker \varphi_i \cong \operatorname{im} \varphi_i$), we may extend it to a map

$$\overline{\psi_i} \colon \overline{\operatorname{im} \varphi_i} \to \overline{V_i/\ker \varphi_i} = \overline{V_i}/\overline{\ker \varphi_i},$$

which obviously is the inverse to $\overline{\varphi_i} \colon \overline{V_i}/\overline{\ker \varphi_i} \to \overline{\operatorname{im} \varphi_i}$ showing the desired equality $\overline{\operatorname{im} \varphi_i} = \operatorname{im} \overline{\varphi_i}$.

Coming back to our functor $G \mapsto G \bar{\otimes} \mathbb{C}$, we may now prove the following lemma:

Lemma 5.16. *Let*

$$\dots \longrightarrow G_i \xrightarrow{\varphi_i} G_{i+1} \xrightarrow{\varphi_{i+1}} G_{i+2} \longrightarrow \dots$$

be an exact sequence of topological groups and continuous maps, which is in addition topologically exact, i.e., for all $i \in \mathbb{Z}$ the from φ_i induced map $G_i/\ker \varphi_i \to \operatorname{im} \varphi_i$ is an isomorphism of topological groups.

Then

$$\dots \longrightarrow G_i \bar{\otimes} \mathbb{C} \longrightarrow G_{i+1} \bar{\otimes} \mathbb{C} \longrightarrow G_{i+2} \bar{\otimes} \mathbb{C} \longrightarrow \dots$$

with the induced maps is an exact sequence of complete topological vector spaces, which is also topologically exact.

Proof. We first tensor with $\mathbb{C}$ (without the completion afterwards). This is known to be an exact functor and our sequence also stays topologically exact. To see this last claim, we need the following fact about tensor products: if $\varphi \colon M \to M'$ and $\psi \colon N \to N'$ are surjective, then the kernel of $\varphi \otimes \psi \colon M \otimes M' \to N \otimes N'$ is the submodule given by

$$\ker(\varphi \otimes \psi) = (\iota_M \otimes 1)\big((\ker \varphi) \otimes N\big) + (1 \otimes \iota_N)\big(M \otimes (\ker \psi)\big),$$

where $\iota_M \colon \ker \varphi \to M$ and $\iota_N \colon \ker \psi \to N$ are the inclusion maps. We will suppress the inclusion maps from now on to shorten the notation.

We apply this with the map $\varphi \colon M \to M'$ being the quotient map $G_i \to G_i/\ker \varphi_i$ and $\psi \colon N \to N'$ being the identity $\mathrm{id} \colon \mathbb{C} \to \mathbb{C}$ to get

$$\ker(\varphi_i \otimes \mathrm{id}) = (\ker \varphi_i) \otimes \mathbb{C}.$$

Since we have $(\operatorname{im} \varphi_i) \otimes \mathbb{C} = \operatorname{im}(\varphi_i \otimes \mathrm{id})$, we get that $\varphi \otimes \mathrm{id} \colon G_i \otimes \mathbb{C} \to G_i \otimes \mathbb{C}$ induces an algebraic isomorphism $(G_i/\ker \varphi_i) \otimes \mathbb{C} \to \operatorname{im} \varphi_i \otimes \mathbb{C}$. But this has now an inverse map

given by tensoring the inverse of $G_i/\ker\varphi_i \to \operatorname{im}\varphi_i$ with $\mathrm{id}\colon \mathbb{C} \to \mathbb{C}$. So the isomorphism $(G_i/\ker\varphi_i) \otimes \mathbb{C} \cong \operatorname{im}\varphi_i \otimes \mathbb{C}$ is also topological.

Now we apply the discussion before the lemma to show that the completion of this new sequence is still exact and also topologically exact. □

To show $K_u^*(M) \bar{\otimes} \mathbb{C} \cong H^*_{u,\mathrm{dR}}(M)$ it remains to construct Mayer–Vietoris sequences with continuous maps in them (we need this since in constructing the completed tensor product with $\mathbb{C}$ we have to pass to the completion of the spaces and without continuity of the maps in both the Mayer–Vietoris sequences for uniform K-theory and for uniform de Rham cohomology we would not be able to conclude that the squares are still commutative). If we recall from the proof of Proposition 5.11 how we get the boundary maps in the Mayer–Vietoris sequence for uniform de Rham cohomology, we see that we must construct a continuous split to a certain surjective map. That this is possible is content of the following lemma and the proof uses the boundedness of the coefficient functions of uniform forms and all their derivatives.

Lemma 5.17. *The sequence*

$$0 \to C_b^\infty(\Omega^p(U_K \cup U_{k+1})) \to C_b^\infty(\Omega^p(U_K)) \oplus C_b^\infty(\Omega^p(U_{k+1})) \to C_b^\infty(\Omega^p(U_K \cap U_{k+1})) \to 0$$

is exact by a continuous split

$$C_b^\infty(\Omega^p(U_K \cap U_{k+1}) \to C_b^\infty(\Omega(U_K)) \oplus C_b^\infty(\Omega^p(U_{k+1})).$$

So we get a Mayer–Vietoris sequence with continuous maps for uniform de Rham cohomology as needed. Now we have to discuss the existence of the Chern character $K_u^*(O \subset M) \to H^*_{u,\mathrm{dR}}(O)$. Recall from Definition 4.34 that we defined $K_u^*(O \subset M)$ as $K_{-*}(C_u(O,d))$, where (O,d) is the metric space O equipped with the subspace metric derived from the metric space M. But for the definition of the Chern character we have to pass to a smooth subalgebra of $C_u(O,d)$. This will be of course $C_b^\infty(O) \subset C_u(O,d)$, which is a local C^*-algebra. It remains to argue why it is a dense subalgebra, because the argument from the proof of Lemma 4.7 does not work for O. So let $f \in C_u(O,d)$ be given. Then we know from Lemma 4.35 that there is a bounded, uniformly continuous extension F of f to all of M. And now we use Lemma 4.7 to approximate F by functions from $C_b^\infty(M)$, which will give us by restriction to O an approximation of f by functions from $C_b^\infty(O)$. So we get an interpretation of $K_u^*(O \subset M)$ by vector bundles of bounded geometry over O and may define the Chern character $K_u^*(O \subset M) \to H^*_{u,\mathrm{dR}}(O)$.

The last thing that we have to discuss is the small ambiguity in extending the maps $K_u^*(O \subset M) \otimes \mathbb{C} \to H^*_{u,\mathrm{dR}}(O)$ to $K_u^*(O \subset M) \bar{\otimes} \mathbb{C}$. It occurs because the target $H^*_{u,\mathrm{dR}}(O)$ is not necessarily Hausdorff. What we have to make sure is that the extensions we choose in the Mayer–Vietoris argument for the subsets U_k, resp. U_K, do match up, i.e., produce at the end commuting squares in the comparison of the two Mayer–Vietoris sequences via the Chern characters.

So we have finally discussed everything that we need in order to prove

$$K_u^*(M) \bar{\otimes} \mathbb{C} \cong H^*_{u,\mathrm{dR}}(M).$$

Proving the homological version $K_*^u(M) \bar{\otimes} \mathbb{C} \cong H_*^{u,\mathrm{dR}}(M)$ is also such a Mayer–Vietoris argument. But for spin^c manifolds there is an easier argument by combining the cohomological result $K_u^*(M) \bar{\otimes} \mathbb{C} \cong H^*_{u,\mathrm{dR}}(M)$ with Theorem 5.18 by noting that taking the wedge product with $\operatorname{ind}(D)$ is an isomorphism on bounded de Rham cohomology, and furthermore using Poincaré duality between uniform K-theory and uniform K-homology (Theorem 4.28), resp., between bounded de Rham cohomology and uniform de Rham homology (Theorem 5.9).

5.4 Local index formulas

In this section we assemble everything that we had up to now into local index theorems.

Let M be a Riemannian manifold without boundary. We denote by DM the disk bundle $\{\xi \in T^*M \colon \|\xi\| \le 1\}$ of its cotangent bundle and by $SM = \partial DM$ its boundary, i.e., $SM = \{\xi \in T^*M \colon \|\xi\| = 1\}$. If M has bounded geometry, we may equip DM with a Riemannian metric such that it also becomes of bounded geometry[80] and $DM \to M$ becomes a Riemannian submersion. It follows that SM will also have bounded geometry. What follows will be independent of the concrete choice of metric on DM. Though we have discussed in Section 4 only uniform K-theory for manifolds without boundary, one can of course define more generally relative uniform K-theory and discuss it for manifolds with boundary and of bounded geometry.

Let $P \in \mathrm{U}\Psi\mathrm{DO}^k(E)$ be a symmetric, elliptic and graded pseudodifferential operator. Recall from Definition 2.35 of ellipticity that the principal symbol $\sigma(P^+)$, viewed as a section of $\operatorname{Hom}(\pi^*E^+, \pi^*E^-) \to T^*M$, where $\pi \colon T^*M \to M$ is the cotangent bundle, is invertible outside a uniform neighbourhood of the zero section $M \subset T^*M$ and satisfies a certain uniformity condition. Then the well-known clutching construction gives us the following symbol class of P:

$$\sigma_P := [\pi^*E^+, \pi^*E^-; \sigma(P)] \in K_u^0(DM, SM).$$

If P is ungraded, then its symbol $\sigma(P) \colon \pi^*E \to \pi^*E$, where $\pi \colon SM \to M$ denotes now the unit sphere bundle of M, is a uniform, self-adjoint automorphism. So it gives a direct sum decomposition $\pi^*E = E^+ \oplus E^-$, where E^+ and E^- are spanned fiberwise by the eigenvectors belonging to the positive, resp. negative, eigenvalues of $\sigma(P)$, and we get an element

$$[E^+] \in K_u^0(SM).$$

Now we define in the ungraded case the symbol class of P as

$$\sigma_P := \delta[E^+] \in K_u^1(DM, SM),$$

where $\delta \colon K_u^0(SM) \to K_u^1(DM, SM)$ is the boundary homomorphism of the 6-term exact sequence associated to (DM, SM). References for this construction in the compact case are, e.g., [BD82, Section 24] and [APS76, Proposition 3.1].

[80] Though we do not have defined bounded geometry for manifolds with boundary, there is an obvious one (demanding bounds not only for the curvature tensor of M but also for the second fundamental form of the boundary of M, and demanding the injectivity radius being uniformly positive not only for M but also for ∂M with the induced metric). See [Sch01] for a further discussion.

Applying the Chern character and integrating over the fibers we get in both the graded and ungraded case $\pi_! \operatorname{ch} \sigma_P \in H^*_{b,\mathrm{dR}}(M)$ and then the index class of P is defined as

$$\operatorname{ind}(P) := (-1)^{\frac{n(n+1)}{2}} \pi_! \operatorname{ch} \sigma_P \wedge \operatorname{Td}(M) \in H^*_{b,\mathrm{dR}}(M),$$

where $n = \dim M$.

Let M be a spin^c manifold of bounded geometry and let us denote by D the Dirac operator associated to the spin^c structure of M. Note that it is m-multigraded, where m is the dimension of the manifold M, and so defines an element in $K^u_m(M)$. Therefore cap product with D is a map $K^*_u(M) \to K^u_{m-*}(M)$, which is an isomorphism (as we have shown in Section 4.4). We also have Poincaré duality $H^*_{b,\mathrm{dR}}(M) \to H^{u,\mathrm{dR}}_{m-*}(M)$, and the content of our local index theorem for uniform twisted Dirac operators is to put these duality maps into a commutative diagram using the homological Chern character on the right hand side and on the cohomology side the index class of the twisted operator.

Theorem 5.18 (Local index theorem for twisted uniform Dirac operators)**.** *Let M be an m-dimensional spin^c manifold of bounded geometry and without boundary. Denote the associated Dirac operator by D.*

Then we have the following commutative diagram:

$$\begin{tikzcd}
K^*_u(M) \arrow[r, "\bullet \cap [D]", "\cong"'] \arrow[d, "\operatorname{ch}(\bullet)\wedge \operatorname{ind}(D)"'] & K^u_{m-*}(M) \arrow[d, "\alpha_* \circ \operatorname{ch}^*"] \\
H^*_{b,\mathrm{dR}}(M) \arrow[r, "\cong"'] & H^{u,\mathrm{dR}}_{m-*}(M)
\end{tikzcd}$$

Proof. This follows from the calculations carried out by Connes and Moscovici in their paper [CM90, Section 3] by noting that the computations also apply in our case where we have bounded geometry and the uniformity conditions. Note that there the cyclic cocycles are defined using expressions in the operators e^{-tD^2}. To translate to the definition of the homological Chern character that we use, see, e.g., [GBVF00, Section 10.2]. $\square$

Remark 5.19. The uniform homological Chern character $\alpha_* \circ \operatorname{ch}^* \colon K^u_*(M) \dashrightarrow H^{u,\mathrm{dR}}_*(M)$ is a priori not well-defined (to be more precise, it is defined on uniformly finitely summable Fredholm modules and it is a priori not clear whether it descends to classes and even whether every class may be represented by a uniformly finitely summable module). But using Poincaré duality between uniform K-homology and uniform K-theory and the above local index theorem, we see that it is a posteriori well-defined for spin^c manifolds. Note that since D is a Dirac operator, it defines a uniformly finitely summable Fredholm module, and therefore also all its twists given by taking the cap product with uniform K-theory classes are uniformly finitely summable.

That the uniform homological Chern character is well-defined for every manifold M of bounded geometry is content of Corollary 5.21.

Let $P \in \mathrm{U\Psi DO}^k(E)$ be a symmetric and elliptic pseudodifferential operator (graded or ungraded). We have a uniform K-homology class $[P] \in K^u_*(M)$ by Theorem3.36 and

therefore, if P is in addition uniformly finitely summable, we may compare the homology class $(\alpha_* \circ \operatorname{ch}^*)(P) \in H_*^{u,\mathrm{dR}}(M)$ with $\operatorname{ind}(P) \in H^*_{b,\mathrm{dR}}(M)$ using the Poincaré duality map (which is now not necessarily an isomorphism since we do no longer assume that M is orientable). That they are equal is the content of the next theorem.

Theorem 5.20 (Local index formula for uniform pseudodifferential operators)**.** *Let M be a Riemannian manifold of bounded geometry and without boundary. Let $P \in \mathrm{U\Psi DO}^k(E)$ be a symmetric and elliptic pseudodifferential operator of positive order acting on a vector bundle $E \to M$ of bounded geometry, and let P be uniformly finitely summable*[81]*.*

Then $\operatorname{ind}(P) \in H^*_{b,\mathrm{dR}}(M)$ is mapped by the duality map $H^*_{b,\mathrm{dR}}(M) \to H_*^{u,\mathrm{dR}}(M)$ to the class $(\alpha_* \circ \operatorname{ch}^*)(P) \in H_*^{u,\mathrm{dR}}(M)$.

Proof. This follows from Theorem 5.18 by the same sequence of reduction steps as in the proof of [CM90, Theorem 3.9]: first we pass to the orientation cover of M, then we take a suitable product with S^1 if M is odd-dimensional, and at last we use the fact that for orientable, even-dimensional manifolds uniform K-homology is spanned modulo 2-torsion by generalized signature operators. This last fact will follow from Theorem 5.23. ☐

Corollary 5.21. *The uniform homological Chern character*

$$\alpha_* \circ \operatorname{ch}^* \colon K_*^u(M) \to H_*^{u,\mathrm{dR}}(M)$$

is well-defined for every manifold M of bounded geometry and without boundary.

Proof. If M is spin^c we know by Poincaré duality that every class $[x] \in K_*^u(M)$ may be represented by a uniformly finitely summable Fredholm module and by the above Theorem 5.20 we conclude that $(\alpha_* \circ \operatorname{ch}^*)([x])$ is independent of the concrete choice of such a representative. (This was already mentioned in Remark 5.19.)

If M is not spin^c we have to use the same sequence of reduction steps as in the proof of Theorem 5.20. ☐

Remark 5.22. The condition in the above Theorem 5.20 that P has to be uniformly finitely summable may actually be dropped. The statement then would be that $(\alpha_* \circ \operatorname{ch}^*)([P])$ is the dual of the class $\operatorname{ind}(P) \in H^*_{b,\mathrm{dR}}(M)$. This makes sense since we now know that the uniform homological Chern character $K_*^u(M) \to H_*^{u,\mathrm{dR}}(M)$ is well-defined. But the problem now is that in order to compute $(\alpha_* \circ \operatorname{ch}^*)([P])$ we would have to replace P by some other operator P' which defines the same uniform K-homology class as P but which is uniformly finitely summable (so that we may compute the Chern–Connes character). This seems to be a task which is not easily carried out in practice.

Connes and Moscovici work in [CM90] with so-called *θ-summable Fredholm modules* which are more general than finitely summable modules. So defining an appropriate version of *uniformly θ-summable Fredholm modules* we could certainly prove the above Theorem 5.20 for them and therefore weakening the condition on P that it has to be uniformly finitely summable.

[81] This means that P defines a uniformly finitely summable Fredholm module, i.e., $\chi(P)$ is uniformly finitely summable for some normalizing function χ.

Let us state now the Thom isomorphism theorem in the form that we need for the proof of the above Theorem 5.20.

Theorem 5.23 (Thom isomorphism)**.** *Let M be a Riemannian spin^c manifold of bounded geometry and without boundary.*

Then the principal symbol of the Dirac operator associated to the spin^c structure of M constitutes an orientation class for $K_u^*(DM, SM)$, i.e., it implements the isomorphism $K_u^*(M) \cong K_u^*(DM, SM)$.

If M is only oriented (i.e., not necessarily spin^c) and even-dimensional, the principal symbol of the signature operator of M constitutes an orientation class modulo 2-torsion.

Proof. The usual proof as found in, e.g., [LM89, Appendix C], works in our case analogously. Note that for the proof of [LM89, Theorem C.7] we have to cover M by such subsets as we used in our proof of Poincaré duality (see Lemma 4.33) since only in this case we have shown that we have a Mayer–Vietoris sequence for uniform K-theory. □

In [CM90, Theorem 3.9] the local index theorem was written using an index pairing with compactly supported cohomology classes. We can of course do the same also here in our uniform setting and the statement is at first glance the same.[82] But the difference is that due to the uniformness we have an additional continuity statement.

Corollary 5.24. *Let $[\varphi] \in H^k_{c,\mathrm{dR}}(M)$ be a compactly supported cohomology class and define the analytic index $\operatorname{ind}_{[\varphi]}(P)$ as in [CM90].[83] Then we have*

$$\operatorname{ind}_{[\varphi]}(P) = \int_M \operatorname{ind}(P) \wedge [\varphi]$$

and this pairing is continuous, i.e., $\int_M \operatorname{ind}(P) \wedge [\varphi] \leq \|\operatorname{ind}(P)\|_\infty \cdot \|[\varphi]\|_1$, where $\|\cdot\|_\infty$ denotes the sup-seminorm on $H^{m-k}_{b,\mathrm{dR}}(M)$ and $\|\cdot\|_1$ the L^1-seminorm on $H^k_{c,\mathrm{dR}}(M)$.

Remark 5.25. Though it may seem that the above corollary is in some sense equivalent to Theorem 5.20, it is in fact not. It is weaker in the following way: in the case of a non-compact M (the case of interest in this paper) the bounded de Rham cohomology $H^*_{b,\mathrm{dR}}(M)$ usually contains elements of seminorm $= 0$ and due to the bounded of the above pairing we see that we can not detect those elements by that pairing.

5.5 Index pairings on amenable manifolds

In the last section we proved the local index theorems for uniform operators. The goal of this section is to use these local formulas to compute certain global indices of such operators over amenable manifolds.

[82] Remember that we have another choice of universal constants than Connes and Moscovici, i.e., in our statement they are not written since they are incorporated in the definition of the homological Chern character.

[83] Note that $\operatorname{ind}_{[\varphi]}(P)$ is analytically defined and may be computed (up to the universal constant that we have incorporated into the definition of $\alpha_* \circ \mathrm{ch}^*$) as $\langle (\alpha_* \circ \mathrm{ch}^*)(P), [\varphi] \rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing between uniform de Rham homology and compact supported cohomology.

So in this section we assume that our manifold M is *amenable*, i.e., that it admits a Følner sequence. We will need such a sequence in order to construct the index pairings.

Definition 5.26 (Følner sequences)**.** Let M be a manifold of bounded geometry. A sequence of compact subsets $(M_i)_i$ of M will be called a *Følner sequence*[84] if for each $r > 0$ we have

$$\frac{\operatorname{vol} B_r(\partial M_i)}{\operatorname{vol} M_i} \xrightarrow{i\to\infty} 0.$$

A Følner sequence $(M_i)_i$ will be called a *Følner exhaustion*, if $(M_i)_i$ is an exhaustion, i.e., $M_1 \subset M_2 \subset \ldots$ and $\bigcup_i M_i = M$.

Note that if M admits a Følner sequence, then it is always possible to construct a Følner exhaustion for M (the author did this construction in its full glory in his thesis [Eng14, Lemma 2.38]).

For example, Euclidean space $\mathbb{R}^m$ is amenable, but hyperbolic space $\mathbb{H}^{m\geq 2}$ not. Furthermore, if M has subexponential volume growth[85], then M is amenable (this is proved in [Roe88a, Proposition 6.2]). Note that the converse to this last statement is wrong, i.e., there are examples of amenable spaces with exponential volume growth. Further examples of amenable manifolds arise from the theorem that the universal covering $\widetilde{M}$ of a compact manifold M is amenable (if equipped with the pull-back metric) if and only if the fundamental group $\pi_1(M)$ is amenable (this is proved in [Bro81]).

Let M^m be a connected and oriented manifold of bounded geometry. Then there is a duality isomorphism $H^m_{b,\mathrm{dR}}(M) \cong H_0^{\mathrm{uf}}(M;\mathbb{R})$, where the latter denotes the uniformly finite homology of Block and Weinberger. This isomorphism is mentioned in the remark at the end of Section 3 in [BW92] and proved explicitely in [Why01, Lemma 2.2].[86] Since we have the characterization [BW92, Theorem 3.1] of amenability stating that M is amenable if and only if $H_0^{\mathrm{uf}}(M) \neq 0$, we therefore also have a characterization of it via bounded de Rham cohomology. We are going to discuss this now a bit more closely.

First we introduce the following notions:

Definition 5.27 (Closed at infinity, [Sul76, Definition II.5])**.** A Riemannian manifold M is called *closed at infinity* if for every function f on M with $0 < C^{-1} < f < C$ for some $C > 0$, we have $[f \cdot dM] \neq 0 \in H^m_{b,\mathrm{dR}}(M)$ (where dM denotes the volume form of M and $m = \dim M$).

Definition 5.28 (Fundamental classes, [Roe88a, Definition 3.3])**.** A *fundamental class* for the manifold M is a positive linear functional $\theta\colon \Omega_b^m(M) \to \mathbb{R}$ such that $\theta(dM) \neq 0$ and $\theta \circ d = 0$.

[84] In [Roe88a, Definition 6.1] such sequences were called *regular*.

[85] This means that there exists a point $x_0 \in M$ such that for all $p > 0$ we have $e^{-pr}\operatorname{vol}(B_r(x_0)) \xrightarrow{r\to\infty} 0$.

[86] Alternatively, we could use the Poincaré duality isomorphism $H^i_{b,\mathrm{dR}}(M) \cong H^\infty_{m-i}(M;\mathbb{R})$ which is proved in [AB98, Theorem 4], where $H^\infty_{m-i}(M;\mathbb{R})$ denotes simplicial L^∞-homology and M is triangulated according to Theorem 4.31, and then use the fact that $H_0^\infty(M;\mathbb{R}) \cong H_0^{\mathrm{uf}}(M;\mathbb{R})$ under this triangulation (for this we need the assumption that M is connected).

If we are given a Følner sequence for M, we can construct a fundamental class for M out of it; this is done in [Roe88a, Propositions 6.4 & 6.5].[87] But admitting a fundamental class implies that M is closed at infinity.[88] This means especially $H^m_{b,\mathrm{dR}}(M) \neq 0$. But since this is isomorphic to $H_0^{\mathrm{uf}}(M;\mathbb{R})$, we conclude that the latter does also not vanish. So M is amenable, i.e., admits a Følner sequence, and so we are back at the beginning of our chain. Let us summarize this:

Proposition 5.29. *Let M be a connected, orientable manifold of bounded geometry.*
Then the following are equivalent:

- *M admits a Følner sequence,*
- *M admits a fundamental class and*
- *M is closed at infinity.*

We know that the universal cover $\widetilde{M}$ of a compact manifold M is amenable if and only if $\pi_1(M)$ is amenable. If this is the case, then we may construct fundamental classes that respect the structure of $\widetilde{M}$ as a covering space:

Proposition 5.30 ([Roe88a, Proposition 6.6])**.** *Let M be a compact Riemannian manifold, denote by $\widetilde{M}$ its universal cover equipped with the pull-back metric, and let $\pi_1(M)$ be amenable.*
Then $\widetilde{M}$ admits a fundamental class θ with the property

$$\theta(\pi^*\alpha) = \int_M \alpha$$

for every top-dimensional form α on M and where $\pi\colon \widetilde{M} \to M$ is the covering projection.

At last, let us state just for the sake of completeness the relation of amenability to the linear isoparametric inequality.

Proposition 5.31 ([Gro81b, Subsection 4.1])**.** *Let M be a connected and orientable manifold of bounded geometry.*
Then M is not amenable if and only if $\mathrm{vol}(R) \le C \cdot \mathrm{vol}(\partial R)$ *for all $R \subset M$ and a fixed constant $C > 0$.*

We can also detect amenability of M using the K-theory of the uniform Roe algebra $C_u^*(\Gamma)$ of a quasi-lattice $\Gamma \subset M$.[89] Recall that one possible definition for the uniform Roe algebra $C_u^*(\Gamma)$ is the norm closure of the *-algebra of all finite propagation operators in $\mathfrak{B}(\ell^2(\Gamma))$ with uniformly bounded coefficients.

[87] If $(M_i)_i$ is a Følner sequence, then the linear functionals $\theta_i(\alpha) := \frac{1}{\mathrm{vol}\, M_i}\int_{M_i} \alpha$ are elements of the dual of $\Omega_b^m(M)$ and have operator norm $= 1$. Now take θ as a weak-* limit point of $(\theta_i)_i$. The Følner condition for $(M_i)_i$ is needed to show that θ vanishes on boundaries.

[88] Just use the positivity of the fundamental class θ: $\theta(f \cdot dM) \ge \theta(C^{-1} \cdot dM) = C^{-1} \cdot \theta(dM) \neq 0$.

[89] see Definition 3.14

Proposition 5.32 ([Ele97]). *Let M be a manifold of bounded geometry and let $\Gamma \subset M$ be a uniformly discrete quasi-lattice.*

Then M is amenable if and only if $[1] \neq [0] \in K_0(C_u^*(\Gamma))$, where $[1] \in K_0(C_u^*(\Gamma))$ is a certain distinguished class.

The reason why we stated the above proposition is that it introduces functionals on $K_0(C_u^*(\Gamma))$ associated to Følner sequences that we will need in the definition of our index pairings. So let us recall Elek's argument: Let $(\Gamma_i)_i$ be a Følner sequence in Γ and let $T \in C_u^*(\Gamma)$. Then we define a bounded sequence indexed by i by $\frac{1}{\#\Gamma_i}\sum_{\gamma\in\Gamma_i} T(\gamma,\gamma)$. Choosing a linear functional $\tau \in (\ell^\infty)^*$ associated to a free ultrafilter on $\mathbb{N}$[90] we get a linear functional θ on $C_u^*(\Gamma)$. The Følner condition for $(\Gamma_i)_i$ is needed to show that θ is a trace, i.e., descends to $K_0(C_u^*(\Gamma))$. Then $\theta([1]) = 1$ and $\theta([0]) = 0$ for the distinguished classes $[1], [0] \in K_0(C_u^*(\Gamma))$.

Let us finally come to the definition of the index pairings that we are interested in.

Definition 5.33. Let M be a manifold of bounded geometry, let $(M_i)_i$ be a Følner sequence for M and let $\tau \in (\ell^\infty)^*$ a linear functional associated to a free ultrafilter on $\mathbb{N}$. Denote the resulting functional on $K_0(C_u^*(\Gamma))$ by θ, where $\Gamma \subset M$ is a quasi-lattice.[91]

Then we define for $p = 0, 1$ an index pairing

$$\langle\cdot,\cdot\rangle_\theta \colon K_u^p(M) \otimes K_p^u(M) \to \mathbb{R}$$

by the formula

$$\langle[x],[y]\rangle_\theta := \theta\big(\mu_u([x]\cap[y])\big),$$

where $\mu_u \colon K_*^u(M) \to K_*(C_u^*(\Gamma))$ is the rough assembly map (see Section 3.5).

If P is a graded, elliptic uniform pseudodifferential operator acting on a graded vector bundle E, then there is a nice way of computing the above index pairing of P with the trivial bundle $[\mathbb{C}] \in K_u^0(M)$: Recall from Corollary 2.45 that if $f \in \mathcal{S}(\mathbb{R})$ is a Schwartz function, then $f(P) \in \mathrm{U\Psi DO}^{-\infty}(E)$, i.e., $f(P)$ is a quasilocal smoothing operator. So by Proposition 2.14 it has a uniformly bounded integral kernel $k_{f(P)}(x,y) \in C_b^\infty(E \boxtimes E^*)$. Now we choose an even function $f \in \mathcal{S}(\mathbb{R})$ with $f(0) = 1$ and get a bounded sequence

$$\frac{1}{\operatorname{vol} M_i}\int_{M_i} \operatorname{tr}_s k_{f(P)}(x,x)\; dM(x),$$

where tr_s denotes the super trace (recall that E is graded), on which we may evaluate τ. This will coincide with the pairing $\langle[\mathbb{C}],P\rangle_\theta$ and is exactly the analytic index that was defined by Roe in [Roe88a] for Dirac operators. For details why this will coincide with $\langle[\mathbb{C}],P\rangle_\theta$ the reader may consult, e.g., the author's Ph.D. thesis [Eng14, Section 2.8].

Let us now define the pairing between uniform de Rham cohomology and uniform de Rham homology. So let $\beta \in C_b^\infty(\Omega^p(M))$ and $C \in \Omega_p^u(M)$, fix an $\epsilon > 0$ and choose for

[90] That is, if we evaluate τ on a bounded sequence, we get the limit of some convergent subsequence.

[91] Note that here we first have to construct from the Følner sequence $(M_i)_i$ for M a corresponding Følner sequence $(\Gamma_i)_i$ for Γ.

every $M_i \subset M$ from a Følner sequence for M a smooth cut-off function $\varphi_i \in C_c^\infty(M)$ with $\varphi_i|_{M_i} \equiv 1$, $\operatorname{supp} \varphi_i \subset B_\epsilon(M_i)$ and such that for all $k \in \mathbb{N}_0$ the derivatives $\nabla^k \varphi_i$ are bounded in sup-norm uniformly in the index i. Then $\varphi_i \beta \in W^{\infty,1}(\Omega^p(M))$ and therefore we may evaluate C on it. The sequence $\frac{1}{\operatorname{vol} M_i} C(\varphi_i \beta)$ will be bounded and so we may apply $\tau \in (\ell^\infty)^*$ to it. Due to the Følner condition for $(M_i)_i$ this pairing will descend to (co-)homology classes.

Definition 5.34. Let M be a manifold of bounded geometry, let $(M_i)_i$ be a Følner sequence for M and let $\tau \in (\ell^\infty)^*$ a linear functional associated to a free ultrafilter on $\mathbb{N}$.

For every $p \in \mathbb{N}_0$ we define a pairing

$$\langle \cdot, \cdot \rangle_{(M_i)_i, \tau} \colon H^p_{u,\mathrm{dR}}(M) \otimes H_p^{u,\mathrm{dR}}(M) \to \mathbb{C}$$

by evaluating τ on the sequence $\frac{1}{\operatorname{vol} M_i} C(\varphi_i \beta)$, where $\beta \in H^p_{u,\mathrm{dR}}(M)$, $C \in H_p^{u,\mathrm{dR}}(M)$ and the cut-off functions φ_i are chosen as above.

Note that this pairing is, similar to the pairing from Corollary 5.24, continuous against the topologies on $H^*_{u,\mathrm{dR}}(M)$ and on $H_*^{u,\mathrm{dR}}(M)$.

Recall that in the usual case of compact manifolds the index pairing for K-theory and K-homology is compatible with the Chern-Connes character, i.e., $\langle [x], [y] \rangle = \langle \operatorname{ch}([x]), \operatorname{ch}([y]) \rangle$ for $[x] \in K^*(M)$ and $[y] \in K_*(M)$. The same also holds in our case here.

Lemma 5.35. *Denote by* $\operatorname{ch} \colon K_u^*(M) \to H^*_{u,\mathrm{dR}}(M)$ *the Chern character on uniform* K*-theory and by* $(\alpha_* \circ \operatorname{ch}^*) \colon K_*^u(M) \to H_*^{u,\mathrm{dR}}(M)$ *the one on uniform* K*-homology.*

Then we have

$$\big\langle [x], [y] \big\rangle_\theta = \big\langle \operatorname{ch}([x]), (\alpha_* \circ \operatorname{ch}^*)([y]) \big\rangle_{(M_i)_i, \tau}$$

for all $[x] \in K_u^p(M)$ *and* $[y] \in K_p^u(M)$.

The last thing that we need is the compatibility of the index pairings with cup and cap products. This is clear by definition for the index pairing for uniform K-theory with uniform K-homology, and for the pairing for uniform de Rham cohomology with uniform de Rham homology it is stated in the following lemma.

Lemma 5.36. *Let* $[\beta] \in H^p_{u,\mathrm{dR}}(M)$, $[\gamma] \in H^q_{u,\mathrm{dR}}(M)$ *and* $[C] \in H_{p+q}^{u,\mathrm{dR}}(M)$. *Then we have*

$$\langle [\beta] \wedge [\gamma], [C] \rangle_{(M_i)_i, \tau} = \langle [\beta], [\gamma] \cap [C] \rangle_{(M_i)_i, \tau}.$$

So combining the above two lemmas together with the results of Section 5.4 we finally arrive at our desired index theorem for amenable manifolds which generalizes Roe's index theorem from [Roe88a] from graded generalized Dirac operators to arbitrarily graded, elliptic, symmetric uniform pseudodifferential operators.

Corollary 5.37. *Let* M *be a manifold of bounded geometry and without boundary, let* $(M_i)_i$ *be a Følner sequence for* M *and let* $\tau \in (\ell^\infty)^*$ *be a linear functional associated to a free ultrafilter on* $\mathbb{N}$. *Denote the from the choice of Følner sequence and functional* τ *resulting functional on* $K_0(C_u^*(\Gamma))$ *by* θ, *where* $\Gamma \subset M$ *is a quasi-lattice.*

Then for both $p \in \{0,1\}$, every $[P] \in K_p^u(M)$ for P a p-graded, elliptic, symmetric uniform pseudodifferential operator over M, and every $u \in K_u^p(M)$ we have

$$\langle u, [P] \rangle_\theta = \langle \operatorname{ch}(u) \wedge \operatorname{ind}(P), [M] \rangle_{(M_i)_i, \tau}.$$

Remark 5.38. The right hand side of the formula in the above corollary reads as

$$\tau\Big(\frac{1}{\operatorname{vol} M_i} \int_{M_i} \operatorname{ch}(u) \wedge \operatorname{ind}(P) \Big)$$

and this is continuous against the sup-seminorm on $H^m_{b,\mathrm{dR}}(M)$ with $m = \dim(M)$, i.e.,

$$\langle u, [P] \rangle_\theta \le \| \operatorname{ch}(u) \wedge \operatorname{ind}(P) \|_\infty.$$

So, again as in Remark 5.25, we see that with this pairing we can not detect operators P whose index class $\operatorname{ind}(P) \in H^*_{b,\mathrm{dR}}(M)$ has sup-seminorm $= 0$ in every degree.

Note that it seems that from the results in [Sul76, Part II.§4] it follows that every element in $H^m_{b,\mathrm{dR}}(M)$ of non-zero sup-seminorm may be detected by a Følner sequence (i.e., the dual space $\overline{H}^*_{b,\mathrm{dR}}(M)$ of the reduced bounded de Rham cohomology[92] is spanned by Følner sequences). So the difference between the statement of the above corollary and Theorem 5.20 lies, at least in top-degree, exactly in the fact that Theorem 5.20 also encompasses all the elements of sup-seminorm $= 0$.

Example 5.39. Let us discuss quickly an example that shows that we indeed may lose information by passing to the reduced bounded de Rham cohomology groups. Roe showed in [Roe88b, Proposition 3.2] that if M^m is a connected spin manifold of bounded geometry, then $\langle \hat{A}(M), [M] \rangle_{\cdot,\cdot} = 0$ for any choice of Følner sequence and suitable functional τ if M has non-negative scalar curvature, and later Whyte showed in [Why01, Theorem 2.3] that $\hat{A}(M) = [0] \in H^m_{b,\mathrm{dR}}(M)$ under these assumptions. So any connected spin manifold M of bounded geometry with $\hat{A}(M) \neq [0] \in H^m_{b,\mathrm{dR}}(M)$ but $\hat{A}(M) = [0] \in \overline{H}^m_{b,\mathrm{dR}}(M)$ can not have non-negative scalar curvature, but this is not detected by the reduced group. In [Why01] it is also shown how one can construct examples of manifolds whose $\hat{A}$-genus vanishes in the reduced but not in the unreduced group.

6 Final remarks and open questions

In this final section we will comment on some questions that remained open, resp., arise out of the theory developed in this article. Most of them are of the form that one wants to generalize certain results known for compact manifolds to the uniform, non-compact setting, but the proofs used in the compact case to not immediately generalize.

But there are also some questions remaining that are not of the above type like the comparison of the coarse and rough assembly map, or like the question whether quasilocal operators may be approximated by operators of finite propagation.

[92] Reduced bounded de Rham cohomology is defined as $\overline{H}^*_{b,\mathrm{dR}}(M) := H^*_{b,\mathrm{dR}}(M)/\overline{[0]}$, i.e., as the Hausdorffication of bounded de Rham cohomology.

6.1 Abundance of derivatives

This is a remark similar to the analogous one in [Roe88b, Section 6.1]. We have been working here with manifolds and vector bundles of bounded geometry in the sense that the curvature tensor and all its derivatives must be bounded. But it seems that a lot of the results presented here do not need boundedness of all the derivatives.

One can see this in the uniform (co-)homology theories that we introduced in this paper. Uniform K-theory may be either defined by using the C^*-algebra of uniformly continuous functions $C_u(X)$ which makes sense on every metric space X, or by using $C_b^\infty(M)$, where we now have high regularity. Bounded de Rham cohomology is isomorphic to uniform de Rham cohomology (here we have high regularity) and also isomorphic to L^∞-simplicial cohomology when we triangulate M as a simplicial complex of bounded geometry using Theorem 4.31 (which is in itself an examples of the interplay between low regularity and high regularity, see Remark 4.32). And uniform de Rham homology is isomorphic to L^∞-simplicial homology.

On the other hand, we have given in this paper definitions of the Chern characters using only the high regularity pictures of the (co-)homology theories. But the author does not see how to give corresponding definitions in the other pictures which do not refer to smoothness.

Question 6.1. *How to define the uniform Chern characters $K_u^*(L) \to H_\infty^*(L)$ and $K_*^u(L) \to H_*^\infty(L)$ for a simplicial complex L of bounded geometry equipped with the metric derived from barycentric coordinates?*

6.2 Geometric picture of uniform K-homology

Baum and Douglas defined in [BD82] a geometric version of K-homology, where the cycles are spin^c manifolds with a vector bundle over them together a map into the space. This geometric picture is quite important for the understanding of index theory and so the question is whether we also have something similar for uniform K-homology.

Question 6.2. *Is there a geometric picture of uniform K-homology that coincides with the analytic one on simplicial complexes of bounded geometry?*

A complete proof that geometric K-homology coincides on finite CW-complexes with analytic K-homology was given by Baum, Higson and Schick in [BHS07]. But this proof relies on a comparison of these theories with topological K-homology, i.e., with the homology theory defined by the K-theory spectrum. So their proof does unfortunately not generalize to our uniform setting; at least not directly.

6.3 Comparing assembly maps and positive scalar curvature

Analogous to the non-uniform setting, the rough assembly map $\mu_u \colon K_*^u(M) \to K_*(C_u^*(M))$ provides obstructions against the existence of uniformly positive scalar curvature metrics, i.e., $\mu_u([M]) = [0] \in K_m(C_u^*(M))$ if the manifold M^m is spin and has uniformly positive

scalar curvature. This obstruction is a priori stronger than the one from the coarse assembly map, since the comparison map $K_*(C^*_u(M)) \to K_*(C^*(M))$ may not be injective.

Question 6.3. *Is there an example of a spin manifold M^m of bounded geometry, such that $\mu_u([M]) \neq [0] \in K_m(C^*_u(M))$, but $\mu([M]) = [0] \in K_m(C^*(M))$?*

6.4 Quasilocal operators and the uniform Roe algebra

In the definition of uniform pseudodifferential operators we used for the $(-\infty)$-part of them quasilocal smoothing operators, whereas the rough Baum–Connes assembly map goes into the K-theory of the uniform Roe algebra which is defined as the closure of the finite propagation operators with uniformly bounded coefficients. A priori the definition of quasilocal operators is more general than being in the closure of the finite propagation operators, but one is tempted to conjecture that the notions actually coincide, i.e., that every quasilocal operator is approximable by finite propagation ones. As far as the author knows this question is still open. Results concerning the interplay between large scale geometry of a metric space and certain properties of its uniform Roe algebra suggest that it might be necessary to add an additional assumption on the space (like having finite asymptotic dimension or Property A) in order to show equivalence of this two notions.

Question 6.4. *Defining the uniform Roe algebra to consist of the quasilocal operators with uniformly bounded coefficients, will it coincide with the usual definition, i.e., is every quasilocal operator approximable by finite propagation ones? Might it be that we need additional large scale geometric assumptions on the metric space in order for this to hold?*

The class of uniform pseudodifferential operators defined in this article is in the following sense connected to the above: assume that we would have defined this class in such a way that the $(-\infty)$-part would be an operator which is in the Fréchet closure[93] of the finite propagation smoothing operators. Then the results of Section 2.4 would give a direct connection to the uniform Roe algebra. Indeed, we would then be able to conclude $\overline{\mathrm{U\Psi DO}^{-\infty}(E)} = \overline{\mathrm{U\Psi DO}^{-1}(E)} = C^*_u(E)$, where $C^*_u(E)$ is the uniform Roe algebra of E, i.e., the closure of the finite propagation, uniformly locally compact operators on E. So there would be some merit in defining uniform pseudodifferential operators due to this direct relation to the uniform Roe algebra, though of course we could also just change the definition of the uniform Roe algebra to quasilocal operators in order to relate it to the current definition of uniform pseudodifferential operators.

But if we would do the above, i.e., changing the definition from quasilocal to approximable by finite propagation operators, there would be one piece of information missing that we do have by using quasilocal operators: Recall that in the analysis of uniform pseudodifferential operators Lemma 2.43 was the main technical ingredient which led, e.g., to Corollary 2.45 stating that if f is a Schwartz function, then $f(P) \in \mathrm{U\Psi DO}^{-\infty}(E)$ for P an elliptic and symmetric pseudodifferential operator of positive order. But the

[93]That is to say, in the closure with respect to the family of norms $(\|\cdot\|_{-k,l}, \|\cdot^*\|_{-k,l})_{k,l\in\mathbb{N}}$, where $\|\cdot\|_{-k,l}$ denotes the operator norm $H^{-k}(E) \to H^l(E)$.

author does not know whether Lemma 2.43 would also hold for the changed definition, i.e., whether under the conditions of that lemma the operator e^{itP} would be approximable in the needed operator norm by finite propagation operators.

Question 6.5. *Does Lemma 2.43 specialize to the statement that if the $(-\infty)$-part of P is in the Fréchet closure of the finite propagation smoothing operators, then e^{itP} is approximable by finite propagation operators of order k in the operator norms $\|\cdot\|_{lk,lk-k}$ for all $l \in \mathbb{Z}$?*

6.5 Analysis of uniform pseudodifferential operators

We know that the principal symbol map σ^k induces an isomorphism of vector spaces $\mathrm{U\Psi DO}^{k-[1]}(E,F) \cong \mathrm{Symb}^{k-[1]}(E,F)$ for all $k \in \mathbb{Z}$ and vector bundles E, F of bounded geometry. For the case $k=0$ and $E=F$ we furthermore know from Proposition 2.23 that $\mathrm{U\Psi DO}^{0-[1]}(E)$ is a commutative algebra, and σ^0 will be an isomorphism of algebras.

In the case that the manifold M is compact, it is known that σ^0 is continuous against the quotient norm[94] on $\Psi\mathrm{DO}^{0-[1]}(E)$ and therefore σ^0 will induce an isomorphism of C^*-algebras $\overline{\Psi\mathrm{DO}^{0-[1]}(E)} \cong \overline{\mathrm{Symb}^{0-[1]}(E)}$.

Question 6.6. *Let M be a non-compact manifold of bounded geometry. Does σ^0 induce an isomorphism of C^*-algebras $\overline{\mathrm{U\Psi DO}^{0-[1]}(E)} \cong \overline{\mathrm{Symb}^{0-[1]}(E)}$?*

To show this we would have to compare the quotient norms on $\mathrm{U\Psi DO}^{0-[1]}(E)$ and on $\mathrm{Symb}^{0-[1]}(E)$. The first to prove similar results in the compact case were Seeley in [See65, Lemma 11.1] and Kohn and Nirenberg in [KN65, Theorem A.4], and two years later Hörmander provided in [Hör67, Theorem 3.3] a proof of this for his class $S^0_{\rho,\delta}$ with $\delta < \rho$ of pseudodifferential operators of order 0. Maybe one of these proofs generalizes to our case of uniform pseudodifferential operators on open manifolds.

The main technical part in the proof of Theorem 3.36 that a uniform pseudodifferential operator defines a class in uniform K-homology was to show that the operator $\chi(P)$ is uniformly pseudolocal for χ a normalizing function. In Proposition 2.33 we have shown that pseudodifferential operators of order 0 are automatically uniformly pseudolocal. So if we could show that the operator $\chi(P)$ is a pseudodifferential operator of order 0, the proof of Theorem 3.36 would follow immediately.

Question 6.7. *Under which conditions on the function f (or the operator P) will be $f(P)$ again a uniform pseudodifferential operator?*

For a compact manifold M there are quite a few proofs that under certain conditions functions of pseudodifferential operators are again pseudodifferential operators: the first one to show such a result was seemingly Seeley in [See67], where he proved it for complex powers of elliptic classical pseudodifferential operators. It was then extended by Strichartz in [Str72] from complex powers to symbols in the sense of Definition 2.47, and

[94] Which is induced from the operator norm on $\Psi\mathrm{DO}^0(E) \subset \mathfrak{B}(L^2(E))$. Since for M compact we have $\overline{\Psi\mathrm{DO}^{-1}(E)} = \mathfrak{K}(L^2(E))$, the quotient norm on $\Psi\mathrm{DO}^{0-[1]}(E)$ is called the *essential norm*.

from classical operators to all of Hörmander's class $S^k_{1,0}(M)$. And last, let us mention the result [DS99, Theorem 8.7] of Dimassi and Sjöstrand for h-pseudodifferential operators in the semi-classical setting.

Now if we want to establish similar results in our setting, we get quite fast into trouble: e.g., the proof of Strichartz does not generalize to non-compact manifolds. He crucially uses that on compact manifolds we may diagonalize elliptic operators, which is not at all the case on non-compact manifolds (consider, e.g., the Laplace operator on Euclidean space). Looking for a proof that may be generalized to the non-compact setting, we stumble over Taylor's result from [Tay81, Chapter XII]. There he proves a result similar to Strichartz' but with quite a different proof, which may be possibly generalized to non-compact manifolds. An evidence for this is given by Cheeger, Gromov and Taylor in [CGT82, Theorem 3.3], since this is exactly the result that we want to prove for our pseudodifferential operators, but in the special case of the operator $\sqrt{-\Delta}$, and their proof is a generalization of the one from the above cited book of Taylor. So it seems quite reasonable that we may probably extend the result of Cheeger, Gromov and Taylor to all pseudodifferential operators in our sense.

Let us briefly explain what this has to do with the geometric optics equation: this equation is treated by Taylor in [Tay81, Chapter VIII] for a compact manifold M and it is one of the main ingredients in his proof that functions of pseudodifferential operators are again pseudodifferential operators. So if we want to extend Taylor's result to our pseudodifferential operators on non-compact manifolds, we will first have to solve the geometric optics equation on them. Since this is probably in itself a more interesting problem than the one about functions of pseudodifferential operators, we get a strong motivation for executing the above discussed ideas.

There might be also another approach to the above problem about functions of uniform pseudodifferential operators. In their articles [Bea77] and [Ueb88] Beals and Ueberberg gave characterizations of pseudodifferential operators via certain mapping properties of these operators from the Schwartz space to its dual. As a corollary they derived that the inverse, if it exists, of a pseudodifferential operator of order 0 is again a pseudodifferential operator.

Question 6.8. *Does there exists a similar characterization of uniform pseudodifferential operators on manifolds of bounded geometry as the one in [Bea77] and [Ueb88] by Beals and Ueberberg?*

6.6 Index theorem on non-compact manifolds with boundary

In the compact case there is a generalization of the Atiyah–Singer index theorem to manifolds with boundary involving the η-invariant. This version of the index theorem for compact manifolds with boundary is called the Atiyah–Patodi–Singer index theorem and was introduced in [APS75].

Of course the question whether such a theorem may also be proven in the non-compact case immediately arises.

Question 6.9. *Is there a version of the, e.g., global index theorem for amenable manifolds, for manifolds of bounded geometry and with boundary? What would be the corresponding generalization of the η-invariant?*

A nice prove of the index theorem for manifolds with boundary for Dirac operators was given by Melrose in [Mel93]. He invented the b-calculus, a calculus for pseudodifferential operators on manifolds with boundary, and derived the Atiyah–Patodi–Singer index theorem from it via the heat kernel approach. So it would be desirable to extend his b-calculus to open manifolds with boundary (in the same way as we extended the calculus of pseudodifferential operators to open manifolds in a fruitful way) and then prove the extension of the Atiyah–Patodi–Singer index theorem to manifolds with boundary and of bounded geometry. Note that Roe's proof of his index theorem for open manifolds does also rely on the heat kernel approach, i.e., there is a real chance that we may generalize Melrose's proof to open manifolds with boundary and of bounded geometry.

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